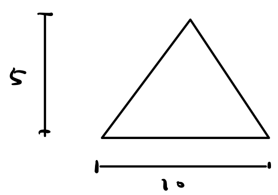
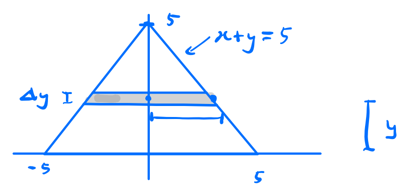
§ 8.1 Areas and Volumes

Today we'll discuss setting up integrals that represent areas and volumes. We'll use the idea of slicing the regions into thin pieces.

Example Use horizontal slices to set up a definite integral for the area of the following isosceles triangle

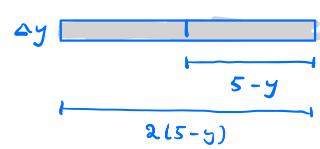




The area of the slice at y units from the base:

width × height

$= 2(5-y) \times \Delta y$



So the total area of n horizontal slices is

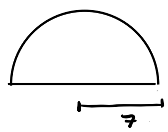
$$\sum_{i=1}^{n} 2(5-y_i)\,\Delta y$$

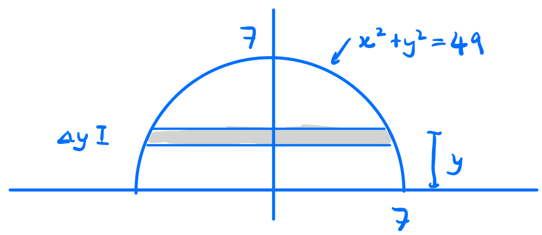
As we let $n \to \infty$ (and $\Delta y \to 0$) this converges to the area of the triangle

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2(5-y_i)\,\Delta y$$

$$= \int_0^5 2(5-y)\,dy = 25$$

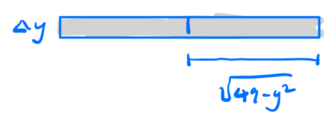
Example Do the same with the semicircle of radius 7





The area of the slice at y units from the base:

width × height



$$= 2\sqrt{49-y^2} \times \Delta y$$

So the total area of n horizontal slices is

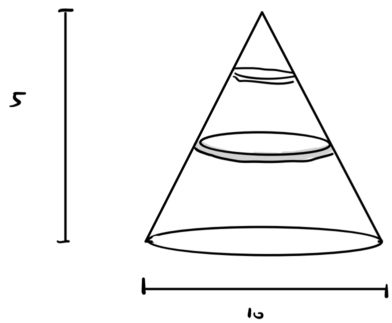
$$\sum_{i=1}^{n} 2\sqrt{49-y^2}\, \Delta y$$

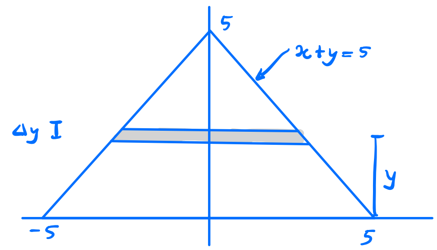
As we let $n \to \infty$ (and $\Delta y \to 0$) this converges to the area of the semicircle

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} 2\sqrt{49-y^2}\, \Delta y$$

$$= \int_0^7 2\sqrt{49-y^2}\, dy = \frac{49\pi}{2}$$

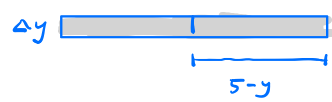
Example: Find the volume of the following cone:





The volume of the slice at y units from the base:

$\pi(\text{radius})^2 \times \text{height}$



$= \pi(5-y)^2 \times \Delta y$

So the total volume of n horizontal slices is

$$\sum_{i=1}^{n} \pi(5-y)^2 \Delta y$$

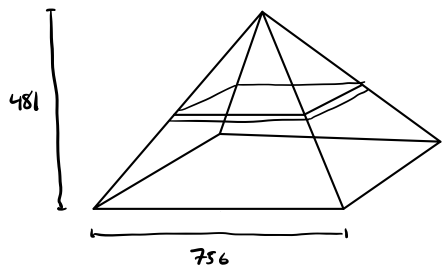
As we let $n \to \infty$ (and $\Delta y \to 0$) this converges to the volume of the cone

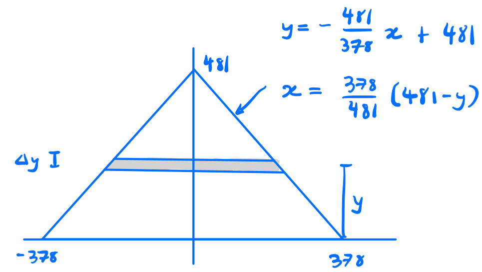
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(5-y)^2 \Delta y$$

$$= \int_0^5 \pi(5-y)^2 \, dy$$

$$= \frac{125}{3}\pi$$

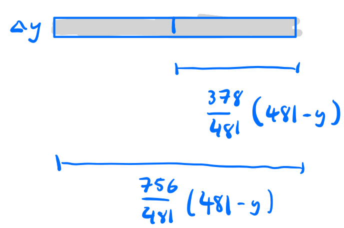
Example Find the volume of the following pyramid with square base





The volume of the slice at y units from the base:

width^2 × height



$$= \left(\frac{756}{481}(481-y)\right)^2 \Delta y$$

So the total volume of n horizontal slices is

$$\sum_{i=1}^{n} \left(\frac{756}{481}(481-y)\right)^2 \Delta y$$

As we let $n \to \infty$ (and $\Delta y \to 0$) this converges to the volume of the cone

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{756}{481}(481-y)\right)^2 \Delta y$$

$$= \int_0^{481} \left(\frac{756}{481}(481-y)\right)^2 dy$$

$$= \left(\frac{756}{481}\right)^2 \int_0^{481} (481-y)^2 \, dy \qquad w = 481-y$$

$$= \left(\frac{756}{481}\right)^2 \frac{(481)^3}{3}$$

$$= \frac{1}{3}(756)^2(481)$$

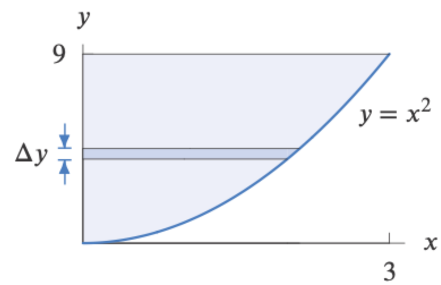
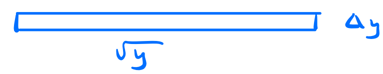


Figure 8.15

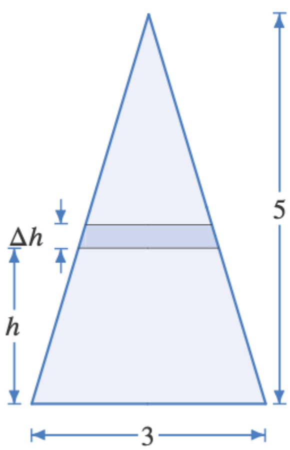


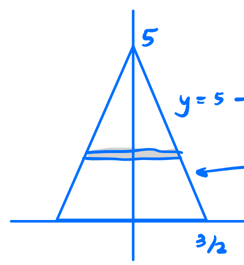
area of slice = $\sqrt{y} \cdot \Delta y$

Riemman sum = $\sum_{i=1}^{n} \sqrt{y_i} \cdot \Delta y$

total area = $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{y_i} \cdot \Delta y$

$= \int_0^9 \sqrt{y}\, dy$





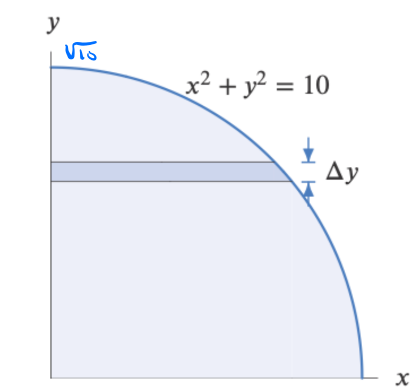
$y = 5 - \frac{5}{3/2} x = 5 - \frac{10}{3} x$

$x = \frac{3}{10}(5-y)$ (equation of line)

area of slice = $2\left(\frac{3}{10}(5-y)\right)\Delta y$

$= \frac{3}{5}(5-y)\Delta y$

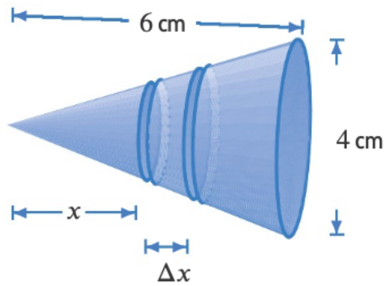
area of triangle = $\int_0^5 \frac{3}{5}(5-y)\, dy$

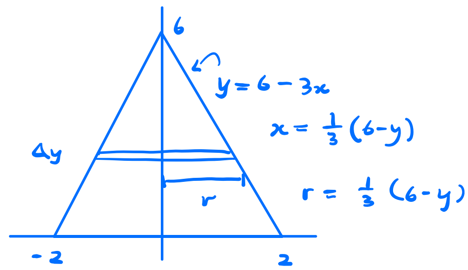


$x = \sqrt{10-y^2}$

area of slice = $\sqrt{10-y^2} \cdot \Delta y$

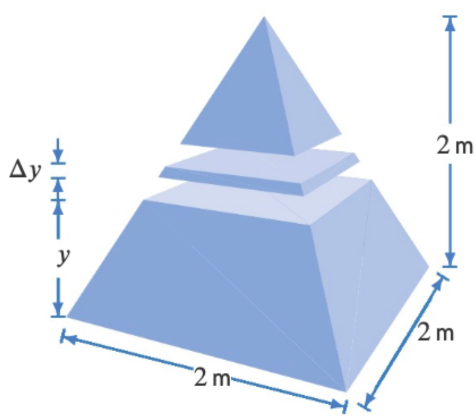
area of region = $\int_0^{\sqrt{10}} \sqrt{10-y^2}\, dy$

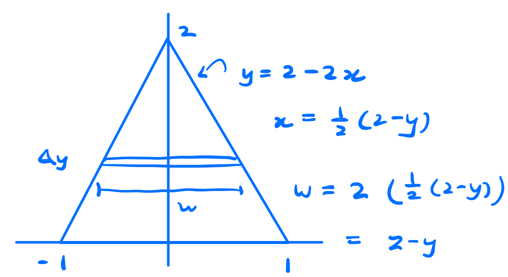




$$\text{volume of slice} = \pi r^2 \Delta y$$
$$= \pi \left(\tfrac{1}{3}(6-y) \right)^2 \Delta y$$

$$\text{total volume} = \int_0^6 \pi \left(\tfrac{1}{3}(6-y) \right)^2 dy$$





$$\text{volume of slice} = w^2 \Delta y$$
$$= (2-y)^2 \Delta y$$

$$\text{total volume} = \int_0^2 (2-y)^2 \, dy$$