6.1 More on substitution

So far we've only seen substitution in the context of indefinite integrals.

Theorem

$$\int_a^b f(g(x))\,g'(x)\,dx \qquad \begin{aligned} u &= g(x) \\ du &= g'(x)\,dx \end{aligned}$$

$$= \int_{g(a)}^{g(b)} f(u)\,du$$

(In words, we change the limits of integration to be in terms of u when doing substitution)

This means we don't have to convert back from u to x after finding antiderivative when doing definite integrals.

Examples

① $\displaystyle\int_0^2 \cos(3x-1)\,dx$

$u = 3x-1$

$du = 3dx$

$\frac{1}{3}du = dx$

$x=0,\ u = 3(0)-1 = -1$

$x=2,\ u = 3(2)-1 = 5$

$$= \int_1^5 \frac{1}{3}\cos u\,du$$

$$= \frac{1}{3}\sin u\,\Big|_1^5 = \frac{1}{3}\sin 5 - \frac{1}{3}\sin 1$$

② $\displaystyle\int_3^4 x e^{x^2}\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x=3,\ u = 3^2 = 9$

$x=4,\ u = 4^2 = 16$

$$= \int_9^{16} \frac{1}{2}e^u\,du$$

$$= \frac{1}{2}e^u\,\Big|_9^{16}$$

$$= \frac{1}{2}\left(e^{16} - e^9\right)$$

③ $\displaystyle\int_4^9 \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$2\,du = x^{-1/2}\,dx$

$= \frac{1}{\sqrt{x}}\,dx$

$x=4,\ u = 4^{1/2} = 2$

$x=9,\ u = 9^{1/2} = 3$

$$= \int_2^3 2\cos u\,du$$

$$= 2\sin u\,\Big|_2^3$$

$$= 2\left(\sin(3) - \sin(2)\right)$$

Problem. Find the following definite integrals using substitution. Make sure to change the limits of integration.

a. $\int_0^2 \sqrt{5x+2}\,dx$

b. $\int_0^1 (8x+2)(2x^2+x)^4\,dx$

c. $\int_0^\pi \cos\left(\frac{x}{2}+\pi\right)\,dx$

d. $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta$

e. $\int_1^4 \frac{e^{x^{1/3}}}{x^{2/3}}\,dx$

(a)

$u = 5x+2$

$du = 5dx$

$\frac{1}{5}du = dx$

$$\frac{1}{5}\int_2^{12}\sqrt{u}\,du = \frac{1}{5}\int_2^{12} u^{1/2}du$$

$$= \frac{2}{15}u^{3/2}\Big|_2^{12}$$

$$= \frac{2}{15}\left(12^{3/2} - 2^{3/2}\right)$$

(b)

$u = 2x^2 + x$

$du = (4x+1)dx$

$2du = (8x+2)\,dx$

$$\int_0^3 2u^4\,du$$

$$= \frac{2}{5}u^5\Big|_0^3 = \frac{2}{5}(3)^5$$

(c)

$u = \frac{x}{2} + \pi$

$du = \frac{1}{2}dx$

$2du = dx$

$$2\int_\pi^{3\pi/2} \cos u\,du$$

$$= 2\sin u\Big|_\pi^{3\pi/2} = -2$$

(d)

$u = -\cos\theta$

$du = \sin\theta\,d\theta$

$$\int_{-1}^0 e^u\,du = e^u\Big|_{-1}^0$$

$$= e^0 - e^{-1} = 1 - e^{-1}$$

(e)

$u = x^{1/3}$

$du = \frac{1}{3}x^{-2/3}\,dx$

$3du = \frac{1}{x^{2/3}}\,dx$

$$3\int_1^{4^{1/3}} e^u\,du$$

$$= 3e^u\Big|_1^{4^{1/3}}$$

$$= 3e^{4^{1/3}} - 3e$$