§ 9.2 Geometric Series, continued

Definition Let $\sum_{k=1}^{\infty} a_k$ be a given series. The nth partial sum of the series is the value

$$S_n = a_1 + a_2 + \cdots + a_n.$$ (sum of the first n terms)

The sequence $(s_1, s_2, s_3, \ldots)$ is called the sequence of partial sums. We say the series $\sum_{k=1}^{\infty} a_k$ converges if the sequence (s_n) converges. Otherwise, we say $\sum_{k=1}^{\infty} a_k$ diverges. When $\lim_{n\to\infty} S_n = L$, we say $\sum_{k=1}^{\infty} a_k = L$.

Example Given a geometric series $a + ax + ax^2 + \cdots$ can we find a formula for the nth partial sum?

$$S_1 = a$$
$$S_2 = a + ax$$
$$S_3 = a + ax + ax^2$$
$$\vdots$$
$$S_n = a + ax + ax^2 + \cdots + ax^{n-1}$$

Note that

$$S_n - xS_n = (a + ax + ax^2 + \cdots + ax^{n-1}) - (ax + ax^2 + ax^3 + \cdots + ax^n)$$
$$= a - ax^n$$

Also $\quad S_n - xS_n = S_n(1-x)$. Therefore

$$S_n = \frac{a - ax^n}{1-x}$$

Notice $\quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} (a + ax + ax^2 + \cdots + ax^{n-1})$

$$= \lim_{n\to\infty} \frac{a - ax^n}{1-x}$$

$$= \frac{a}{1-x} \quad \text{when} \quad |x| < 1.$$

Conclusion: The infinite geometric series $\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \cdots$ converges when $|x| < 1$. In this case it converges to $\frac{a}{1-x}$ and we write $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$. It diverges when $|x| \geq 1$.

Examples: Find the first 5 terms of the sequence of partial sums and the sums of the following series if they converge.

① $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$

$S_1 = 1$, $S_2 = 1 + \frac{1}{2} = 1.5$, $S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75$,

$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$, $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1.9375$

$a = 1$, $x = \frac{1}{2}$

② $6 - 2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \ldots = \frac{6}{1-(-\frac{1}{3})} = 4.5$

$S_1 = 6$, $S_2 = 6-2 = 4$, $S_3 = 6-2+\frac{2}{3} \approx 4.67$

$S_4 = 6-2+\frac{2}{3}-\frac{2}{9} \approx 4.44$, $S_5 = 6-2+\frac{2}{3}-\frac{2}{9}+\frac{2}{27} \approx 4.52$

$a = 6$, $x = -\frac{1}{3}$

③ $1 + 2 + 4 + 8 + 16 + \ldots$

$S_1 = 1$, $S_2 = 1+2 = 3$, $S_3 = 1+2+4 = 7$, $S_4 = 1+2+4+8 = 15, \ldots$

$$S_n = \frac{1-2^n}{1-2} = 2^n - 1$$

$a = 1$, $x = 2$ The series diverges.

Example For which values of z do the following series converge? What do they converge to?

① $1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots$

$$= 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots$$

$$= \frac{1}{1-\frac{z}{2}} = \frac{2}{2-z}$$

Converges when $\left|\frac{z}{2}\right| < 1 \iff -1 < \frac{z}{2} < 1$

$\iff -2 < z < 2$

$\iff z \in (-2, 2)$

"element of"

open interval (endpoints excluded)

(2) $\sum_{n=0}^{\infty} \frac{1}{2}\left(1-\frac{z}{2}\right)^n = \frac{1}{2} + \frac{1}{2}\left(1-\frac{z}{2}\right) + \frac{1}{2}\left(1-\frac{z}{2}\right)^2 + \cdots$

$$= \frac{\frac{1}{2}}{1-\left(1-\frac{z}{2}\right)} = \frac{1}{z}$$

Converges when $\left|1-\frac{z}{2}\right| < 1 \iff -1 < 1-\frac{z}{2} < 1$

$\iff -2 < -\frac{z}{2} < 0$

$\iff 2 > \frac{z}{2} > 0$

$\iff 4 > z > 0$

$\iff z \in (0, 4)$

Problem 1. For each of the following, determine whether it is a geometric series. If it is, state the first term, the common ratio between success terms, and whether it converges. If it converges, find the value it converges to; that is, find its sum.

a. $5 - 10 + 20 - 40 + 80 - \cdots$

b. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots$

c. $\frac{1}{3} + \frac{4}{9} + \frac{16}{27} + \frac{64}{81} + \cdots$

d. $\sum_{n=0}^{\infty} \frac{5}{3^{3n}}$

e. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2^n}$

f. $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^{2n+2}}$

g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(a) $a = 5$, $x = -2$, diverges since $|x| \geq 1$

(b) $a = 2$ $x = -\frac{1}{2}$, converges to $\frac{2}{1-(-\frac{1}{2})} = \frac{4}{3}$

(c) $a = \frac{1}{3}$, $x = \frac{4}{3}$, diverges since $|x| \geq 1$

(d) $\sum_{n=0}^{\infty} 5\left(\frac{1}{27}\right)^n$ $a = 5$, $x = \frac{1}{27}$, converges to $\frac{5}{1-\frac{1}{27}} = \frac{135}{26}$

(e) $\sum_{n=2}^{\infty} -\left(\frac{-1}{2}\right)^n$ $a = \frac{-1}{4}$, $x = \frac{-1}{2}$, converges to $\frac{-1/4}{1-(-1/2)} = \frac{-1}{6}$

(f) $\sum_{n=1}^{\infty} \frac{1}{16}\left(\frac{-1}{16}\right)^n$ $a = \frac{-1}{256}$, $x = \frac{-1}{16}$. Converges to $\frac{-1/256}{1-(-1/16)} = \frac{-1}{272}$

(g) not geometric

Problem 2. For each of the following geometric series, find a formula for the sum in terms of z and find the values of z, expressed as an interval, for which the series converges.

a. $\sum_{n=0}^{\infty} 5^n z^n$

b. $z/3 - z^2/3 + z^3/3 - z^4/4 + z^5/5 - \cdots$

c. $\sum_{n=2}^{\infty} (z-4)^n$

d. $8 + 8(6-z) + 8(6-z)^2 + 8(6-z)^3 + \cdots$

(a) $a = 1$, $x = 5z$, converges to $\frac{1}{1-5z}$ when

$|5z| < 1 \iff -1 < 5z < 1 \iff -\frac{1}{5} < z < \frac{1}{5} \iff z \in \left(-\frac{1}{5}, \frac{1}{5}\right)$

(b) $a = \frac{z}{3}$, $x = z$, converges to $\frac{z/3}{1-z} = \frac{z}{3(1-z)}$ when

$z \in (-1, 1)$

(c) $a = (z-4)^2$, $x = z-4$, converges to

$\frac{(z-4)^2}{1-(z-4)} = \frac{(z-4)^2}{5-z}$ when $|z-4| < 1$

$\iff -1 < z-4 < 1$

$\iff 3 < z < 5$

$\iff z \in (3, 5)$

(d) $a = 8$, $x = 6-z$, converges to $\frac{8}{1-(6-z)} = \frac{8}{z-5}$

when $|6-z| < 1 \iff -1 < 6-z < 1$

$\iff -7 < -z < -5$

$\iff 7 > z > 5$

$\iff z \in (5, 7)$