§9.3 Convergence of Series

Recall that given a series $\sum_{n=1}^{\infty} a_n$, its sequence of partial sums is the sequence (s_n) where $s_n = a_1 + a_2 + \cdots + a_n$. If $\lim_{n\to\infty} s_n = L$, then we say $\sum_{n=1}^{\infty} a_n$ converges to L. If $\lim_{n\to\infty} s_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example Do the following series converge? Make some conjectures based on (s_n) and discuss with a partner.

1. $\sum_{n=1}^{\infty} 1$ diverges

2. $\sum_{n=1}^{\infty} (-1)^n$ diverges

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges

1. $(s_n) = (1, 2, 3, 4, \ldots)$

2. $(s_n) = (-1, 0, -1, 0, \ldots)$

3. $(s_n) = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$

Example Do the following sequences converge? Make some conjectures based on your intuition and discuss with a partner.

① $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges

② $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4n+5}{7n^2+3n+2}$ diverges

③ $\displaystyle\sum_{n=1}^{\infty} \left[\left(\tfrac{1}{2}\right)^n + \left(\tfrac{2}{3}\right)^n\right]$ converges

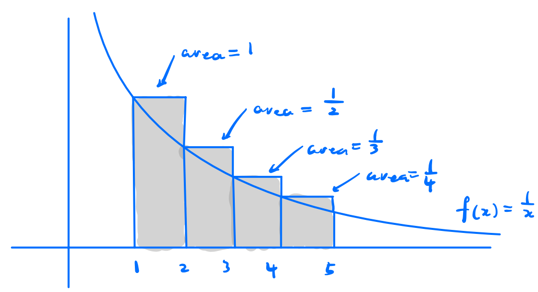
④ $\displaystyle\sum_{n=1}^{\infty} (ca_n + b_n)$ where c is a given constant and $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are given convergent sequences. converges

Theorem (nth term test) If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Warning If $\displaystyle\lim_{n\to\infty} a_n = 0$, it's not necessarily the case that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Theorem If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to A and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges to B, then $\displaystyle\sum_{n=1}^{\infty} [ca_n + b_n]$ converges to $cA + B$.

Example The series $\sum_{n=1}^{\infty} \frac{1}{n}$ (called the **harmonic series**) diverges.

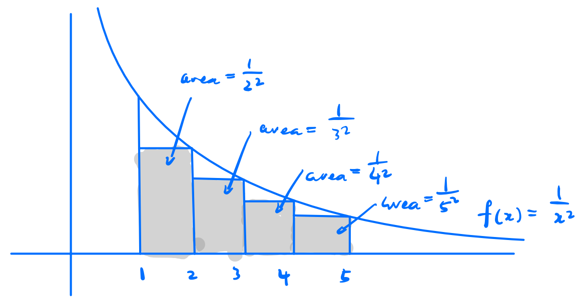


We see in the figure that

$$S_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1).$$

Therefore, since $\lim_{n\to\infty} \ln(n+1) = \infty$, $\lim_{n\to\infty} S_n = \infty$, which means the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.



We see in the figure that

$$s_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 1 + \int_1^n \frac{1}{x^2}\,dx = 2 - \frac{1}{n}$$

And so $\lim_{n\to\infty} s_n \leq 2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Theorem (Integral Test). Suppose $f(x)$ is a function that is positive and decreasing for all $x \geq a$ and $a_n = f(n)$ for all $n \geq a$. Then

1. if $\int_a^\infty f(x)\,dx$ diverges, then $\sum_{n=a}^{\infty} a_n$ diverges
2. if $\int_a^\infty f(x)\,dx$ converges, then $\sum_{n=a}^{\infty} a_n$ converges.

Problem 1. If possible, use the integral test to explain whether the following series converge. If it is not possible to use the integral test, explain why.

a. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

b. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

c. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 0$ is a given constant

d. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

e. $\sum_{n=1}^{\infty} n^3$

(a)
$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx \qquad w = x^2+1$$
$$dw = 2x\,dx$$
$$= \lim_{b\to\infty} \frac{1}{2}\int_2^{b^2+1} \frac{1}{w}\,dw \qquad \frac{1}{2}dw = x\,dx$$
$$= \lim_{b\to\infty} \frac{1}{2} \ln|b^2+1| = \infty$$

So $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges since $\int_1^\infty \frac{x}{x^2+1}\,dx$ diverges

(b)
$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx \qquad w = \ln x$$
$$dw = \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} w^{-2}\,dw$$
$$= \lim_{b\to\infty} w^{-1}\Big|_{\ln b}^{\ln 2}$$
$$= \lim_{b\to\infty} \frac{1}{\ln 2} - \frac{1}{\ln b}$$
$$= \frac{1}{\ln 2}$$

So $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges since $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ converges.

(c) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$.

(d) Not possible since $f(x) = \frac{(-1)^x}{x}$ is not always positive

(e) Not possible since $f(x) = x^2$ is not decreasing.

Problem 2. State what can be concluded from the nth term test for each of the following series.

a. $\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+n}$

b. $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{1+9n^2}}$

c. $\sum_{n=1}^{\infty} 3^n$

d. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

e. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

f. $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{n}\right)$

(a) Since $\lim_{n\to\infty} \frac{n^2+1}{3n^2+n} = \frac{1}{3} \neq 0$, the series $\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+n}$ diverges.

(b) Since $\lim_{n\to\infty} \frac{2n}{\sqrt{1+9n^2}} = \lim_{n\to\infty} \sqrt{\frac{4n^2}{1+9n^2}} = \sqrt{\lim_{n\to\infty} \frac{4n^2}{1+9n^2}} = \sqrt{\frac{4}{9}} = \frac{2}{3} \neq 0$, the series $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{1+9n^2}}$ diverges.

(c) Since $\lim_{n\to\infty} 3^n = \infty \neq 0$, the series diverges

(d) Since $\lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$, the n^{th} term test is inconclusive

(but the series diverges by the integral test)

(e) Since $\lim_{n\to\infty} \frac{1}{n^3} = 0$, the n^{th} term test is inconclusive

(but the series converges by the integral test)

(f) Since $\lim_{n\to\infty} \frac{1}{2^n} + \frac{1}{n} = 0$, the n^{th} term test is inconclusive

(but the series diverges by the integral test)

Problem 3. Consider the series $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$.

a. Find the first three terms s_1, s_2, s_3 of the partial sum sequence (s_n).

b. Find a general formula for s_n. Remember that $\ln(a/b) = \ln a - \ln b$.

c. Use your general formula to compute $\lim_{n\to\infty} s_n$.

d. Does the series converge or diverge?

e. This series is an example of what is called a telescoping series. Where do you think this name comes from?

(a) $s_1 = \ln 2$, $s_2 = \ln 2 + \ln\left(\frac{3}{2}\right) = \ln 3$, $s_3 = \ln 3 + \ln\left(\frac{4}{3}\right) = \ln 4$

(b) $s_n = \ln 2 + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \dots + \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n+1}{n}\right)$

$= \ln 2 + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \dots + (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$

$= \ln(n+1)$

(c) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = \infty$

(d) Since $\lim_{n\to\infty} s_n$ does not exist, the series diverges.

Problem 4. Repeat the previous problem with $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

(a) $s_1 = 1 - \frac{1}{2} = \frac{1}{2}$, $s_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{3}$.

$s_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$.

(b) $s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

(c) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$

(d) The serues converges to 1.