§ 9.4 Tests for Convergence

Last time we introduced the integral test which allowed us to make the following conclusion:

p-series The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (called a p-series) converges when $p > 1$ and diverges when $p \leq 1$.

Comparison Test Suppose $0 \leq a_n \leq b_n$ for all n past a certain value. Then

① if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

② if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Warning if $\sum_{n=1}^{\infty} a_n$ converges, then the test is inconclusive

if $\sum_{n=1}^{\infty} b_n$ diverges, then the test is inconclusive.

Examples Use the comparison test if possible to make a conclusion about the following series:

① $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

② $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$

③ $\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$

④ $\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$

ⓐ Notice $0 \le \frac{1}{n^3+1} \le \frac{1}{n^3}$ for all $n \ge 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (since it's a p-series with $p = 3 > 1$), $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges by the comparison test.

ⓑ Notice $0 \le \frac{n-1}{n^3+3} \le \frac{n}{n^3} = \frac{1}{n^2}$ for all $n \ge 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (since it's a p-series with $p = 2 > 1$), $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$ converges by the comparison test.

(c) Notice $\frac{6n^2+1}{2n^3-1} \geq \frac{6n^2}{2n^3} = \frac{3}{n} \geq 0$ for all $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges (since it's a p-series with $p = 1 \leq 1$), $\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$ diverges by the comparison test.

(d) Notice $\frac{n^2-5}{n^3+n+2} \leq \frac{n^2}{n^3} = \frac{1}{n}$. But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

so the comparison test is inconclusive.

Question What do we do in this last case? It still seems like the series should diverge since $\frac{n^2-5}{n^3+n+2}$ "behaves like" $\frac{1}{n}$ as $n \to \infty$.

Limit Comparison Test Suppose $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge or both converge.

Example Use the Limit Comparison Test to make a conclusion about whether the following series converge.

① $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{9n^2+5n+2}{n^5+4n+7}$

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^3+2n+1}}$

① Observe that

$$\lim_{n\to\infty} \frac{\frac{n^2-5}{n^3+n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2-5}{n^3+n+2} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3-5n}{n^3+n+2}$$

$$= 1 > 0$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$ diverges by the Limit Comparison Test.

(2) Observe that

$$\lim_{n\to\infty} \frac{\frac{9n^2+5n+2}{n^5+4n+7}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{9n^2+5n+2}{n^5+4n+7} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{9n^5+5n^4+2n^3}{n^5+4n+7}$$

$$= 9 > 0$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, $\sum_{n=1}^{\infty} \frac{9n^2+5n+2}{n^5+4n+7}$ converges

by the Limit Comparison Test.

(3) Observe that

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{3n^3+2n+1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{3n^3+2n+1}} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \left(\frac{n^3}{3n^3+2n+1}\right)^{1/2}$$

$$= \left(\frac{1}{3}\right)^{1/2} > 0$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^3+2n+1}}$ converges

by the Limit Comparison Test.

Problem 1. Use the comparison test to explain whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7}$

b. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

c. $\sum_{n=1}^{\infty} \frac{n^2+n+1}{3n^3-1}$

d. $\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

(a) Notice $0 \leq \frac{1}{n^2+4n+7} \leq \frac{1}{n^2}$ for all $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7}$ converges

by the comparison test.

(b) Notice $\frac{1}{\sqrt{n^2-1}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n} \geq 0$ for all $n \geq 2$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7}$ diverges

by the comparison test.

(c) Notice $\frac{n^2+n+1}{3n^3-1} \geq \frac{n^2}{3n^3} = \frac{1}{3n}$ for all $n \geq 1$

Since $\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges, $\sum_{n=1}^{\infty} \frac{n^2+n+1}{3n^3-1}$ diverges

by the comparison test.

(d) Notice $0 \leq \frac{2^n}{5^n+10} \leq \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$ for all $n \geq 1$.

Since $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ converges (it's geometric with $x = \frac{2}{5}$),

$\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$ by the comparison test.

Problem 2. Use the limit comparison test to explain whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{n+1}{5n^3-3n^2-1}$

b. $\sum_{n=1}^{\infty} \frac{2n-1}{5n^2+7n+3}$

c. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{7n^5+3n^2-2}}$

d. $\sum_{n=1}^{\infty} \frac{4n^3}{\sqrt{n^7+3n^2-2}}$

(a) Observe that

$$\lim_{n\to\infty} \frac{\frac{n+1}{5n^3-3n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n+1}{5n^3-3n^2-1} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2}{5n^3-3n^2-1}$$

$$= \frac{1}{5} > 0$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{n+1}{5n^3-3n^2-1}$ converges

by the Limit Comparison Test.

(b) Observe that

$$\lim_{n\to\infty} \frac{\frac{2n-1}{5n^2+7n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n-1}{5n^2+7n+3} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{2n^2-n}{5n^2+7n+3}$$

$$= \frac{2}{5} > 0$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{2n-1}{5n^2+7n+3}$ diverges

by the Limit Comparison Test.

(c) Observe that

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{7n^5+3n^2-2}}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{7n^5+3n^2-2}} \cdot \frac{n^{5/2}}{1}$$

$$= \lim_{n\to\infty} \left(\frac{n^5}{7n^5+3n^2-2}\right)^{1/2}$$

$$= \left(\frac{1}{7}\right)^{1/2} > 0$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{7n^5+3n^2-2}}$ converges by the Limit Comparison Test.

(3) Observe that

$$\lim_{n\to\infty} \frac{\frac{4n^3}{\sqrt{n^7+3n^2-2}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{4n^3}{\sqrt{n^7+3n^2-2}} \cdot \frac{n^{1/2}}{1}$$

$$= \lim_{n\to\infty} \left(\frac{4n^7}{n^7+3n^2-2}\right)^{1/2}$$

$$= 2 > 0$$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, $\sum_{n=1}^{\infty} \frac{4n^3}{\sqrt{n^7+3n^2-2}}$ diverges by the Limit Comparison Test.