§ 9.4 Tests for convergence

We have learned that a series of the form $\sum_{n=1}^{\infty} ax^n$ is called a geometric series. Its key feature is the constant ratio of consecutive terms (common ratio x). Remember that it converges when $|x|<1$. What if we have a series where "eventually" the terms have approximately a constant ratio?

Theorem (Ratio Test) Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

1. if $L<1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. if $L>1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. if $L=1$, the test is inconclusive ($\sum_{n=1}^{\infty} a_n$ might converge or diverge and we need to use other methods to make a conclusion)

Examples Use the ratio test if possible to conclude whether the following series converge.

① $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

Observe that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{2^n}{2^{n+1}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{(n+1)^3}{n^3}$$

$$= \frac{1}{2} < 1$$

By the Ratio Test, the series converges.

② $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

Observe that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$= \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)!}\right|}{\left|\frac{(-1)^n}{n!}\right|}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

By the Ratio Test, the series converges.

③ $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(-8)^n}$

Observe that $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{8^{n+1}} \cdot \frac{8^n}{(2n)!}$$

$$= \frac{1}{8} \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!}$$

$$= \frac{1}{8} \lim_{n\to\infty} (2n+2)(2n+1)$$

$$= \infty > 1.$$

By the Ratio Test, the series diverges.

④ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Observe that $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$= \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{n+1}\right|}{\left|\frac{(-1)^n}{n}\right|}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$=$$

The ratio test is inconclusive. (Later we'll learn it converges.)

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

The same work as the previous example shows the Ratio Test is inconclusive. (We've shown this series diverges using the Integral Test.)

Problem 1. What does the ratio test tell us about each of the following series? If the test is inconclusive, can you use another method to make a conclusion about whether the series converges?

a. $\sum_{n=1}^{\infty} \frac{(-4)^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n3^n}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n!}$

d. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3}$

e. $\sum_{n=1}^{\infty} \frac{n+2}{n^3+n}$

f. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$

(a) Observe that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-4)^{n+1}}{(n+1)!}\right|}{\left|\frac{(-4)^n}{n!}\right|}$$

$$= \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!}$$

$$= 4 \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1.$$

By the Ratio Test the series converges.

(b) Observe that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left| \frac{(-2)^{2n+2}}{(n+1)\,3^{n+1}} \right|}{\left| \frac{(-2)^{2n}}{n \cdot 3^n} \right|}$$

$$= \lim_{n\to\infty} \frac{2^{2n+2}}{2^{2n}} \cdot \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}}$$

$$= \frac{4}{3} \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{4}{3} > 1.$$

By the Ratio Test the series diverges.

(c) Observe that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2(n+1)}$$

$$= 0 < 1$$

By the Ratio Test the series converges.

(d) Observe that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^3} \cdot \frac{n^3}{n!}$$

$$= \lim_{n\to\infty} \frac{n^3(n+1)}{(n+1)^3}$$

$$= \infty$$

By the Ratio Test the series diverges.

(e) Observe that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+3}{(n+1)^3+n+1} \cdot \frac{n^3+n}{n+2}$$

$$= \lim_{n\to\infty} \frac{n^4+3n^3+n^2+3n}{n(n+1)^3+2(n+1)^3+n^2+n+2}$$

$$=1$$

The Ratio Test is inconclusive. However, we can use the Limit Comparison Test: let $b_n = \frac{1}{n^2}$

Then $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+2}{n^3+n} \cdot \frac{n^2}{1}$$

$$= 1 > 0$$

The series converges by the Limit Comparison Test since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it's a p-series with $p>1$).

(f) The Ratio Test is inconclusive. We cannot use the Comparison Tests (limit or regular) since those require the terms to be positive. Later we will learn this series converges by something called the Absolute Convergence Test.

(f) The Ratio Test is inconclusive. We cannot use the Comparison Tests (limit or regular) since those require the terms to be positive. Later we will learn this series converges by something called the Alternating Series Test.