§ 9.4 Tests for Convergence concluded.

We've learned that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is a conditionally convergent series (it converges by the Alternating Series Test but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges).

Fact Soon we'll learn that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

Something weird:

① Start with

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

② Rearrange the terms so that every positive term is followed by two negative terms:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$$

③ Add parentheses and simplify:

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

$$=\left(\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

(4) Notice this is half of what we started with!

$$\left(\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)$$

$$=\frac{1}{2}\ln 2$$

<u>Theorem</u> (Riemann Rearrangement Theorem)

(1) if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, the terms can be rearranged to converge to any value $L \in \mathbb{R}$.

(2) if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, every rearrangement of the terms converges to the same value.