§ 9.5 Power series

A <u>power series</u> is an infinite series that involves a variable x and has the following form:

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n (x-c)^n$$

where $c, a_0, a_1, a_2, \ldots$ are given constants.

<u>Intuition</u>: it's like a polynomial with infinitely many terms.

<u>Goal</u> determine which values of x can be plugged in so that we get a convergent series (ie. determine the domain of this infinite polynomial) for examples.

<u>Example</u> For which values of x does the following power series converge:

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} \quad (\text{does } x=1 \text{ work? } x=3\,?)$$

<u>$x=1$</u> $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges

since it's a geometric series with $r=\frac{1}{2}$

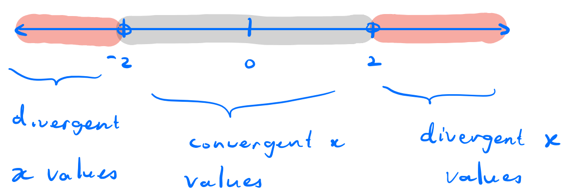
<u>$x=3$</u> $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$, diverges : $r = \frac{3}{2} > 1$

General x $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ converges

when $\left|\frac{x}{2}\right| < 1$, diverges when $\left|\frac{x}{2}\right| \geq 1$

$$\left|\frac{x}{2}\right| < 1 \iff -1 < \frac{x}{2} < 1 \iff -2 < x < 2$$

$$\left|\frac{x}{2}\right| \geq 1 \iff \frac{x}{2} \geq 1 \text{ or } \frac{x}{2} \leq -1 \iff x \geq 2 \text{ or } x \leq -2$$



Terminology $(-2, 2)$ is called the interval of convergence of this power series. And $R = 2$ is called the radius of convergence

Remark When this power series converges, it represents the function $f(x) = \frac{1}{1-\left(\frac{x}{2}\right)}$ (from geometrics series formula).

Example Find the interval of convergence and radius of convergence for the following power series.

① $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n}$

① We use the ratio test, treating x like a constant:

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^n (x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left|\frac{x-1}{3}\right| \cdot \frac{n}{n+1} = \left|\frac{x-1}{3}\right|$$

Series converges when $L<1$, diverges when $L>1$ and test is inconclusive when $L=1$.

$$L<1 \iff \left|\frac{x-1}{3}\right| < 1 \iff -1 < \frac{x-1}{3} < 1$$

$$\iff -3 < x-1 < 3$$

$$\iff -2 < x < 4$$

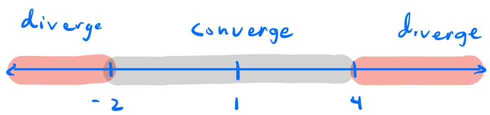
$$L>1 \iff \left|\frac{x-1}{3}\right| > 1 \iff \frac{x-1}{3} > 1 \text{ or } \frac{x-1}{3} < -1$$

$$\iff x-1 > 3 \text{ or } x-1 < -3$$

$$\iff x > 4 \text{ or } x < -2$$

$$L=1 \iff \left|\frac{x-1}{3}\right| = 1 \iff \frac{x-1}{3} = \pm 1$$

$$\iff x-1 = \pm 3 \iff x = -2, 4$$



What happens at endpoints? Use another test:

$\underline{x = -2}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(-3)^n}{n \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n \cdot 3^n}{n \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges, } p = 1$$

$\underline{x = 4}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

converges by alternating series test.

Interval convergence: $(-2, 4]$ Radius of convergence: $R = 3$

don't include endpoint

include endpoint