8.7 Taylor's Theorem

We can use a Taylor polynomial to approximate a function value at a particular x-value.

This brings to mind two questions:

① given n, how large is the error ε in the approximation $p_n(x)$ of $f(x)$?

② given ε, how large should we choose n so that the approximation has error less than ε?

Example Let $f(x) = \cos x$ and let $P_6(x)$ be the Maclaurin polynomial of f of degree 6.

① Estimate $|f(2) - P_6(2)|$

② Find n so that $|f(2) - p_n(2)| < 0.001$

Theorem (Taylor's Theorem) Let $f(x)$ be a function that is $(n+1)$-times differentiable on an interval I that contains c. Then for every x in I,

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{\text{Taylor polynomial } p_n(x) \text{ of degree } n \text{ at } c} + \underbrace{R_n(x)}_{\text{"remainder" or "error"}}$$

and $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}$$ where M is the maximum value of $|f^{(n+1)}(z)|$ over all z between x and c.

Summary A function is equal to its Taylor polynomial of degree n, up to an error that can be approximated using the $(n+1)$st derivative of the function.