§ 7.1 Integration by Substitution

In Calculus I, we learn the chain rule of derivatives:

$$\frac{d}{dx}\left(f(g(x)) \right) = f'(g(x))\, g'(x)$$

Example $\frac{d}{dx}\left(\sin(x^3) \right) = \cos(x^3) \cdot 3x^2$

Our goal today is to use this to do integrals that involve compositions of functions.

Example (1) $\int 3x^2 \cos(x^3)\, dx$

Of course based on our example above, we can see that the antiderivative will be $\sin(x^3) + C$.

But what if we had not done that example above?

We do what is called substitution or change of variables:

$$\int \cos(\underbrace{x^3}_{w}) \cdot \underbrace{3x^2\, dx}_{dw}$$

$w = x^3$

$\frac{dw}{dx} = 3x^2$

$dw = 3x^2 dx$

$$= \int \cos(w)\, dw$$

$$= \sin(w) + C$$

$$= \sin(x^3) + C$$

② $$\int x^3 \sqrt{x^4+5}\, dx$$

$w = x^4 + 5$

$dw = 4x^3\, dx$

$\frac{1}{4} dw = x^3\, dx$

$$= \int \sqrt{w} \cdot \frac{1}{4}\, dw$$

$$= \frac{1}{4} \int w^{1/2}\, dw$$

$$= \frac{1}{4} \cdot \frac{2}{3} w^{3/2} + C$$

$$= \frac{1}{6} (x^4 + 5)^{3/2} + C$$

③ $$\int x e^{x^2}\, dx$$

$w = x^2$

$dw = 2x\, dx$

$\frac{1}{2} dw = x\, dx$

$$= \int \frac{1}{2} e^w\, dw$$

$$= \frac{1}{2} e^w + C$$

$$= \frac{1}{2} e^{x^2}$$

④ $$\int \frac{e^x}{1+e^x}\, dx$$

$w = 1 + e^x$

$dw = e^x\, dx$

$$= \int \frac{1}{w}\, dw$$

$$= \ln|w| + C$$

$$= \ln|1+e^x| + C$$

⑤ $\int e^x \cos^3(e^x)\sin(e^x)\,dx$

$w = \cos(e^x)$

$dw = -\sin(e^x)\cdot e^x\,dx$

$-dw = \sin(e^x)\cdot e^x\,d$

$= \int -w^3\,dw$

$= -\frac{1}{4}w^4 + C$

$= -\frac{1}{4}\cos^4(e^x) + C$

⑥ $\int x\sqrt{x+3}\,dx$

$w = x+3\,,\quad x = w-3$

$dw = dx$

$= \int (w-3)\sqrt{w}\,dw$

$= \int (w-3)w^{1/2}\,dw$

$= \int (w^{3/2} - 3w^{1/2})\,dw$

$= \frac{2}{5}w^{5/2} - 2w^{3/2} + C = \frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C.$

Problem 1. Find the following indefinite integrals using substitution. Identify w and compute $dw = w'(x)\,dx$ to start.

a. $\int x^2 e^{x^3+1}\,dx$

b. $\int x(x^2+3)^2\,dx$

c. $\int \sin x \cos x\,dx$

d. $\int \frac{1}{x \ln x}\,dx$

e. $\int \frac{(\ln x)^2}{x}\,dx$

(a) $\int \frac{1}{3} e^w\,dw = \frac{1}{3} e^w + C$

$= \frac{1}{3} e^{x^3+1} + C$

$w = x^3 + 1$

$dw = 3x^2\,dx$

$\frac{1}{3}dw = x^2\,dx$

(b) $\int \frac{1}{2} w^2\,dw$

$= \frac{1}{6} w^3 + C$

$= \frac{1}{6}(x^2+3)^3 + C$

$w = x^2 + 3$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

(c) $\int w\,dw$

$= \frac{1}{2} w^2 + C$

$= \frac{1}{2}\sin^2 x + C$

$w = \sin x$

$dw = \cos x\,dx$

(d) $\int \frac{1}{w}\,dw$

$= \ln|w| + C$

$= \ln|\ln x| + C$

$w = \ln x$

$dw = \frac{1}{x}\,dx$

(e) $\int w^2\,dw$

$= \frac{1}{3} w^3 + C$

$= \frac{1}{3}(\ln x)^3 + C$

$w = \ln x$

$dw = \frac{1}{x}\,dx$

Problem 2. Find the following indefinite integrals using substitution.

a. $\int e^{4x}\,dx$

b. $\int e^{-x}\,dx$

c. $\int \sin(2x)\,dx$

d. $\int \cos(5-x)\,dx$

e. $\int \frac{3}{2x+1}\,dx$

f. $\int \frac{7}{1-3x}\,dx$

(a) $\int \frac{1}{4} e^{w}\,dw$ $\quad w = 4x$, $dw = 4dx$, $\frac{1}{4}dw = dx$

$= \frac{1}{4} e^{w} + C$

$= \frac{1}{4} e^{4x} + C$

(b) $-\int e^{w}\,dw$ $\quad w = -x$, $dw = -dx$, $-dw = dx$

$= -e^{w} + C$

$= -e^{-x} + C$

(c) $\frac{1}{2}\int \sin w\,dw$ $\quad w = 2x$, $dw = 2dx$, $\frac{1}{2}dw = dx$

$= -\frac{1}{2}\cos w + C$

$= -\frac{1}{2}\cos(2x) + C$

(d) $-\int \cos w\,dw$ $\quad w = 5-x$, $dw = -dx$, $-dw = dx$

$= -\sin w + C$

$= -\sin(5-x) + C$

(e) $= \int \frac{3}{w}\cdot\frac{1}{2}\,dw$ $\quad w = 2x+1$, $dw = 2dx$, $\frac{1}{2}dw = dx$

$= \frac{3}{2}\int \frac{1}{w}\,dw$

$= \frac{3}{2}\ln|w| + C$

$= \frac{3}{2}\ln|2x+1| + C$

(f) $-\frac{7}{3}\int \frac{1}{w}\,dw$ $\quad w = 1-3x$, $dw = -3dx$, $-\frac{1}{3}dw = dx$

$= -\frac{7}{3}\ln|w| + C$

$= -\frac{7}{3}\ln|1-3x| + C$

Problem 3. Find the following indefinite integrals.

a. $\int \sin^6 x \cos x \, dx$

b. $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

c. $\int \tan x \, dx$

d. $\int x(x+1)^{1/3} \, dx$

e. $\int x^2(1+x)^2 \, dx$

(a)

$$\int w^6 \, dw \qquad w = \sin x$$

$$dw = \cos x \, dx$$

$$= \frac{1}{7} w^7 + C$$

$$= \frac{1}{7} \sin^7 x + C$$

(b)

$$w = \sqrt{x}$$

$$2\int \sin w \, dw \qquad dw = \frac{1}{2} x^{-\frac{1}{2}} dx$$

$$= -2\cos w + C \qquad 2\,dw = \frac{1}{\sqrt{x}} dx$$

$$= -2\cos\sqrt{x} + C$$

(c)

$$\int \frac{\sin x}{\cos x} dx \qquad w = \cos x$$

$$dw = -\sin x \, dx$$

$$= \int -\frac{1}{w} dw \qquad -dw = \sin x \, dx$$

$$= -\ln|w| + C$$

$$= -\ln|\cos x| + C$$

(d)

$$\int (w-1) w^{1/3} \, dw \qquad w = x+1, \quad x = w-1$$

$$dw = dx$$

$$= \int (w^{4/3} - w^{1/3}) \, dw$$

$$= \frac{3}{7} w^{7/3} - \frac{3}{4} w^{4/3} + C$$

$$= \frac{3}{7}(x+1)^{7/3} - \frac{3}{4}(x+1)^{4/3} + C$$