Problem 1. Sketch the region bounded by the given curves and then set up integrals to find the volume obtained by rotating the region about the given axis. Consider only the portion of the region in the 1st quadrant if it spans multiple quadrants.

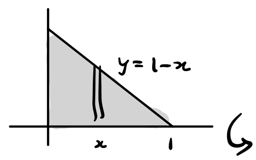
a. $x + y = 1$, $y = 0$, $x = 0$; about the x-axis

b. $y = x^2$, $y = x$; about the y-axis

c. $y = x^2$, $y = 9$, $x = 0$; about the x-axis

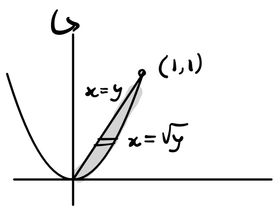
d. $y = x^2$, $y = 9$, $x = 0$; about the y-axis

(a)



$$V = \int_0^1 \pi (1-x)^2 \, dx$$

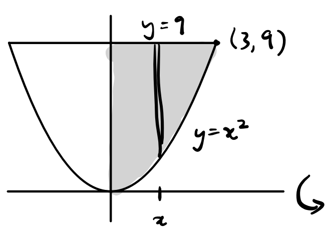
(b)



$$V = \int_0^1 \pi \left[(\sqrt{y})^2 - (y)^2 \right] dy$$

$$= \int_0^1 \pi (y - y^2) \, dy$$

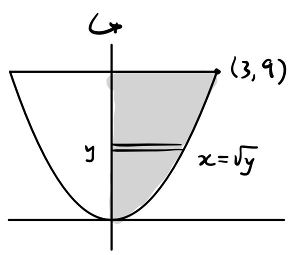
(c)



$$V = \int_0^3 \pi \left[(9)^2 - (x^2)^2 \right] dx$$

$$= \int_0^3 \pi (81 - x^4) \, dx$$

(d)



$$V = \int_0^9 \pi (\sqrt{y})^2 \, dy$$

$$= \int_0^9 \pi y \, dy$$

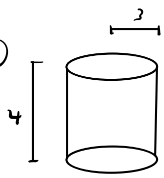
Problem 2. A variety of tanks filled with oil, which has density 800 kg per cubic meter, are given below. Each tank is only filled to half the tank's height. Set up but do not compute an integral for the work performed in pumping all the oil to a height 3 meters above the top of the tank.

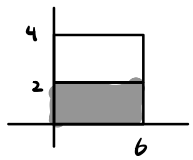
a. An upright circular cylinder with height 4 m and base radius 3 m.

b. A cone whose base circle has radius 3 m and whose height is 10 m, oriented so that its tip is at the bottom of the tank.

c. A pyramid whose base is a square with side length 4 m and whose height is 7 m, oriented so that its tip is at the bottom of the tank.

(a)



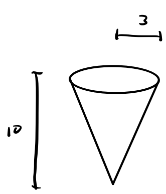


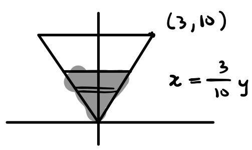
volume of a slice

$= \pi(3)^2 \,\Delta y = 9\pi \,\Delta y$

$$\int_0^2 \underbrace{(800)(9\pi)}_{\text{mass}} \underbrace{(9.8)}_{g} \underbrace{(7-y)}_{d} \,dy$$

(b)



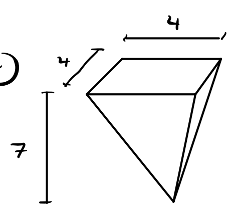


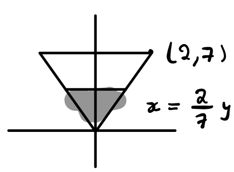
volume of slice

$= \pi\left(\frac{3}{10} y\right)^2 \Delta y$

$$\int_0^5 \underbrace{800\pi\left(\tfrac{3}{10}y\right)^2}_{m} \underbrace{(9.8)}_{g} \underbrace{(13-y)}_{d} \,dy$$

(c)





volume of slice

$= \left(\frac{4}{7} y\right)^2 \Delta y$

$$\int_0^{3.5} \underbrace{800\left(\tfrac{4}{7}y\right)^2}_{m} \underbrace{(9.8)}_{g} \underbrace{(10-y)}_{d} \,dy$$

Problem 3. First, consider a_n below as the nth term of a sequence. State whether the sequence converges and, if so, find its limit. Second, consider a_n as the nth term of a series. State whether the series converges and, if possible, find its sum.

a. $a_n = \frac{n^3+4n^2+3}{8n^4+5n+7}$

b. $a_n = \frac{9^{n+1}}{10^n}$

c. $a_n = \frac{n^4+1}{3n^4+4}$

d. $a_n = n^2 2^{-n}$

e. $a_n = 7$

(a) sequence converges: $\lim_{n\to\infty} \frac{n^3+4n^2+3}{8n^4+5n+7} = 0$

Series diverges, by limit comparison test:

Let $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{8} > 0$.

Since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ diverges.

(b) sequence converges,

$$\lim_{n\to\infty} \frac{9 \cdot 9^n}{10^n} = \lim_{n\to\infty} 9 \cdot \left(\frac{9}{10}\right)^n = 9 \cdot \lim_{n\to\infty} \left(\frac{9}{10}\right)^n = 0$$

Series converges since it's geometric with

$a = 9, x = 9/10$ and converges to $\frac{a}{1-x} = 90.$

(c) sequence converges: $\lim_{n\to\infty} \frac{n^4+1}{3n^4+4} = \frac{1}{3}$

series diverges by nth term test.

(d) Series converges by ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}.$$

sequence converges, $\lim_{n\to\infty} a_n = 0$, since the terms of a convergent series must go to 0.

(e) sequence converges, $\lim_{n\to\infty} a_n = 7$

Series diverges by nth term test

Problem 4. Find the sum of the infinite series

a. $-3+2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+\cdots$

b. $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$

c. $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n}$

(a) This is geometric with $a=-3$, $x=\frac{-2}{3}$. The sum is $\frac{a}{1-x}=\frac{-9}{5}$.

(b) $6+6(0.9)+6(0.9)^2+\cdots$ is geometric with $a=6$, $x=0.9$. The sum is $\frac{a}{1-x}=60$.

(c) $\left(\frac{-3}{4}\right)+\left(\frac{-3}{4}\right)^2+\left(\frac{-3}{4}\right)^3+\cdots$ is geometric with $a=\frac{-3}{4}$, $x=\frac{-3}{4}$. The sum is $\frac{a}{1-x}=\frac{-3}{7}$.

Problem 5. Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges where a_n is given below. State the test used and make sure to justify your use of the test with appropriate details.

a. $a_n = \frac{1}{n\sqrt{n^2+1}}$

b. $a_n = \frac{n!}{5^n}$

c. $a_n = \frac{n}{3n+1}$

d. $a_n = (-1)^n \frac{1}{5n^2+1}$

e. $a_n = (-1)^{n-1} \frac{1}{\sqrt{5n^2+1}}$

f. $a_n = \frac{3}{n+(1.2)^n}$

g. $a_n = \frac{2^n n!}{(n+2)!}$

h. $a_n = \frac{n^{0.1}-1}{n(\sqrt{n}+1)}$

a) converges by limit comparison test:

Let $a_n = \frac{1}{n\sqrt{n^2+1}}$ and $b_n = \frac{1}{n^2}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}}$

$= \lim_{n\to\infty} \sqrt{\frac{n^4}{n^4+n^2}}$

$= \sqrt{1} = 1 > 0$

So $\sum a_n$ converges since $\sum b_n$ converges since it's a p-series with $p = 2 > 1$.

b) diverges by ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$$

c) diverges by n^{th} term test:

$$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0$$

d) Converges by absolute convergence test and comparison (or limit comparison) test:

$$\sum |a_n| = \sum \frac{1}{5n^2+1} \quad \text{and}$$

$$\frac{1}{5n^2+1} \leq \frac{1}{5n^2}$$

and $\sum \frac{1}{5n^2} = \frac{1}{5}\sum \frac{1}{n^2}$ converges since it's a p-series with $p = 2 > 1$

Therefore $\sum |a_n|$ converges by comparison test, which implies $\sum a_n$ converges by absolute convergence test.

[Note: it's ok to use alt. series test here instead, but this technique lets us even say the series converges absolutely]

e) converges by alt. series test:

Let $b_n = \frac{1}{\sqrt{5n^2+1}}$

Then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{5n^2+1}} = 0$

and $b_{n+1} = \frac{1}{\sqrt{5(n+1)^2+1}} \leq \frac{1}{\sqrt{5n^2+1}} = b_n$,

so b_n is decreasing. Then $\sum a_n$ converges by alt. series test.

[Note: this is the only test that can be used]

(f) Converges by comparison test:

Note $a_n = \dfrac{3}{n+(1.2)^n} \le \dfrac{3}{(1.2)^n} = 3\left(\dfrac{1}{1.2}\right)^n$

and $\sum 3\left(\frac{1}{1.2}\right)^n$ is a convergent geometric series, so $\sum a_n$ converges by comparison test

[Note: ratio test is difficult to use here since the limit is a little tricky algebraically]

(g) Diverges by ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(n+3)!}\,\frac{(n+2)!}{2n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(n+1)}{n+3}\right|$$

$$= 2 > 1$$

(h) Converges by comparison (or limit comparison) test:

Let $b_n = \dfrac{1}{n^{1.4}}$. Then

$$a_n = \frac{n^{0.1}-1}{n(\sqrt{n}+1)} \le \frac{n^{0.1}}{n(\sqrt{n})} = \frac{n^{0.1}}{n^{1.5}} = \frac{1}{n^{1.4}} = b_n$$

Since $\sum b_n$ converges, since it's a p-series with $p = 1.4 > 1$, $\sum a_n$ converges too.

Problem 6. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^5+2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{5n+2}}$

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{1/3}+3}$

a) Converges absolutely:

$\sum |a_n| = \sum \frac{n^2}{n^5+2}$, which converges by comparison (or limit comparison) test with $b_n = \frac{1}{n^3}$ since

$$\frac{n^2}{n^5+2} \le \frac{n^2}{n^5} = \frac{1}{n^3}$$

and $\sum b_n$ is a convergent p-series.

b) Converges conditionally

Note $\sum |a_n| = \sum \frac{1}{\sqrt{5n+2}}$ diverges by limit comparison test. If $b_n = \frac{1}{\sqrt{n}}$ then

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{5n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{5n+2}} = \lim_{n\to\infty} \sqrt{\frac{n}{5n+2}} = \sqrt{\frac{1}{5}} > 0.$$

which implies $\sum \frac{1}{\sqrt{5n+2}}$ since $\sum b_n$ is a divergent p-series.

However $\sum a_n$ converges by alt. series test (see problem 4e above).

c) Converges conditionally.

Note $\sum |a_n| = \sum \frac{1}{n^{1/3}+3}$ diverges

by limit comparison test. If $b_n = \frac{1}{n^{1/3}}$ then

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1/3}+3}}{\frac{1}{n^{1/3}}} = \lim_{n\to\infty} \frac{n^{1/3}}{n^{1/3}+3} = 1 > 0$$

which implies $\sum \frac{1}{n^{1/3}+3}$ diverges since

$\sum b_n$ is a divergent p-series.

However $\sum a_n$ converges by alt. series

test since (1) $\lim_{n\to\infty} \frac{1}{n^{1/3}+3} = 0$ and

(2) $\frac{1}{(n+1)^{1/3}+3} \leq \frac{1}{n^{1/3}+3}$