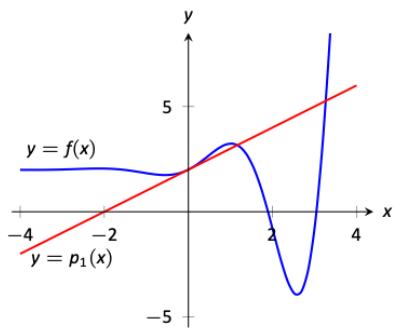


§ 10.1 Taylor Polynomials

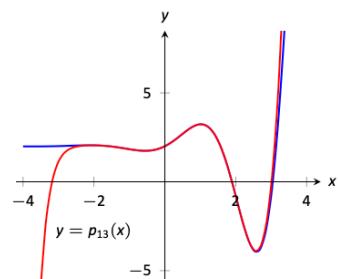
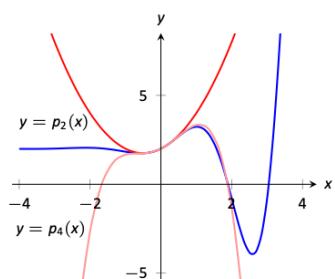
Goal: generalize the idea of tangent line approximation of a function f to quadratic, cubic, or n^{th} degree polynomial approximations



Question what do $f(x)$ and its tangent line

at $x=0$, $p_1(x)$, have in common? $f(0) = p_1(0)$

slope and y-value at $x=0$ $f'(0) = p_1'(0)$



Quadratic, quartic, 13th degree polynomial approximations

Questions ① How should we define a quadratic polynomial so that $p_2(0) = f(0)$, $p'_2(0) = f'(0)$, $p''_2(0) = f''(0)$ (same value at 0, same slope at 0, same concavity at 0)?

Let $p_2(x) = c_0 + c_1 x + c_2 x^2$. Then

$$p'_2(x) = c_1 + 2c_2 x, \quad p''_2(x) = 2c_2.$$

We need to find c_0, c_1, c_2 so that these conditions are satisfied:

$$f(0) = p_2(0) = c_0$$

$$f'(0) = p'_2(0) = c_1$$

$$f''(0) = p''_2(0) = 2c_2 \Rightarrow c_2 = \frac{f''(0)}{2}$$

$$\text{Therefore } p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

is our desired quadratic approximation.

② How do we find a quartic polynomial $p_4(x)$ so that $p_4(0) = f(0)$, $p_4'(0) = f'(0)$, $p_4''(0) = f''(0)$, $p_4'''(0) = f'''(0)$, $p_4^{(4)}(0) = f^{(4)}(0)$?

$$\text{Let } p_4(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

$$p_4'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3$$

$$p_4''(x) = 2c_2 + 6c_3x + 12c_4x^2$$

$$p_4'''(x) = 6c_3 + 24c_4x$$

$$p_4^{(4)}(x) = 24c_4$$

We need to find c_0, c_1, c_2, c_3, c_4 so that:

$$f(0) = p(0) = c_0$$

$$f'(0) = p'(0) = c_1$$

$$f''(0) = p''(0) = 2c_2 \Rightarrow c_2 = \frac{f''(0)}{2}$$

$$f'''(0) = p'''(0) = 6c_3 \Rightarrow c_3 = \frac{f'''(0)}{6}$$

$$f^{(4)}(0) = p_4^{(4)}(0) = 24c_4 \Rightarrow c_4 = \frac{f^{(4)}(0)}{24}$$

$$\text{So } p_4(x) = c_0 + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$

Def Let f be a function whose first n derivatives exist at $x=c$.

(1) The Taylor polynomial of degree n at $x=c$ is

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

(2) The Maclaurin polynomial of degree n of f is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example Find the Maclaurin polynomial of degree n of $f(x) = e^x$ and use $p_5(x)$ to approximate the value of e .

$$f^{(n)}(x) = e^x, \quad f^{(n)}(0) = e^0 = 1,$$

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

$$e = f(1) \approx p_5(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.71667$$

Example Find the Maclaurin polynomial of

degree 10 of $f(x) = \cos x$.

$$f(x) = \cos x \quad f(0) = 1 = f^{(4)}(0) = f^{(8)}(0)$$

$$f'(x) = -\sin x \quad f'(0) = 0 = f^{(5)}(0) = f^{(9)}(0)$$

$$f''(x) = -\cos x \quad f''(0) = -1 = f^{(6)}(0) = f^{(10)}(0)$$

$$f'''(x) = \sin x \quad f'''(0) = 0 = f^{(7)}(0)$$

$$P_{10}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10}$$

Example Find the Taylor polynomial at $x=1$ of

degree 6 of $f(x) = \ln x$.

$$f(x) = \ln x$$

$$f^{(4)}(x) = -\frac{6}{x^4}$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$f'(x) = \frac{1}{x}$$

$$f^{(5)}(x) = \frac{24}{x^5}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f^{(6)}(x) = -\frac{120}{x^6}$$

$$f^{(n)}(1) = (-1)^{n+1}(n+1)!$$

$$f'''(x) = \frac{2}{x^3}$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$P_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6$$

Problem. Find the degree n Maclaurin polynomial for $f(x)$ in each of the following examples.

- a. $f(x) = \sin x, n = 9$
- b. $f(x) = 1/(1 - x), n = 5$
- c. $f(x) = 1/(1 + x), n = 5$
- d. $f(x) = e^{-x}, n = 5$

$$\textcircled{a} \quad P_9(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$

$$\textcircled{b} \quad P_5(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

$$\textcircled{c} \quad P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$$

$$\textcircled{d} \quad P_5(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$$