8.3 p - series and comparison test

Last time (nth term test for divergence)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges

Question If $\lim_{n\to\infty} a_n = 0$, does that mean that $\sum_{n=1}^{\infty} a_n$ converges? (This is called the converse of the theorem above)

Answer we'll soon see that the answer is not necessarily.

Example Let $a_n = \frac{1}{n}$. What is $\lim_{n\to\infty} a_n$?

Does $\sum_{n=1}^{\infty} a_n$ converge or diverge?

Notice $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$, but we'll now

show $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges.

Notice $$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

and this is a divergent series.

To summarize $\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{2} + \dots = +\infty$

which thus implies $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ (ie. it diverges)

Example Let $a_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$. Does $\sum_{n=1}^{\infty} a_n$ converge or diverge?

Let's think about the terms of this series in comparison to our last series

Which is bigger: n or $\sqrt{n}$?

$\frac{1}{n}$ or $\frac{1}{\sqrt{n}}$?

Thus $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots$

$\geq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

and this is a divergent series.

To summarize $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

which implies $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

<u>Exercise</u> Consider, for any $n \geq 1$,

the expressions

$$\frac{1}{n^{0.3}} \geq, \frac{1}{n^{0.4}} \geq, \frac{1}{n^{0.9}} \geq, \frac{1}{n} \geq, \frac{1}{n^{1.1}}$$

Order them from biggest to smallest

(try an example value of n, like $n=2$,

if you're stuck)

What can you conclude about convergence

or divergence of

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{0.3}} \geq, \sum_{n=1}^{\infty} \frac{1}{n^{0.4}} \geq, \sum_{n=1}^{\infty} \frac{1}{n^{0.9}} \geq, \sum_{n=1}^{\infty} \frac{1}{n}}_{\text{all diverge}} \geq, \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}}_{\text{don't know yet!}} ?$$

<u>Theorem</u> (p-series test) Consider the

infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (called a p-series)

① When $p \leq 1$, the series diverges

② When $p > 1$, the series converges