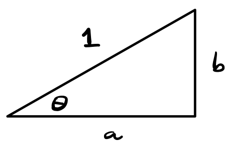
§ 7.4 Trigonometric substitution

Today we'll focus on integrals that involve expressions like $\sqrt{a^2-x^2}$ and a^2+x^2.

First, some trigonometry ideas to keep in mind:



$$\cos\theta = a$$

$$\sin\theta = b$$

$$\tan\theta = \frac{b}{a}$$

Suppose $a^2+b^2=1$.

$$\sin^2\theta + \cos\theta^2 = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

When $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\sin\theta$ is an invertible function whose inverse is $\arcsin(x)$. This function outputs an angle θ in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x.

eg. $\arcsin(1) = \frac{\pi}{2}$

$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Similarly, when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\tan\theta$ is invertible with inverse $\arctan(x)$, which is an angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ whose tangent is x.

Example Find $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$ by substituting $x=\sin\theta$.

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta$$

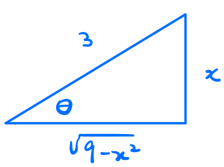
$$= \int d\theta$$

$$= \theta + c = \arcsin(x) + C.$$

Example Find $\displaystyle\int \sqrt{9-x^2}\,dx$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$



$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$= \int \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int \sqrt{9(1-\sin^2\theta)} \cdot 3\cos\theta\, d\theta$$

$$= \int 3\sqrt{1-\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= 9\int \sqrt{1-\sin^2\theta} \cdot \cos\theta\, d\theta$$

$$= 9\int \sqrt{\cos^2\theta} \cdot \cos\theta\, d\theta$$

$$= 9\int \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\left(\sin\theta\cos\theta + \theta\right) + C$$ (found previously using int. by parts)

$$= \frac{9}{2}\left(\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + \arcsin\left(\frac{x}{3}\right)\right) + C$$

$$= \frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C.$$

<u>Example</u> Find $\int \frac{1}{x^2+9}\,dx$

Let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{1}{9\tan^2\theta + 9} \cdot 3\sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{1}{\tan^2\theta+1} \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sec^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int d\theta$$

$$= \frac{1}{3}\theta + C$$

$$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

Problem. Find the following indefinite integrals using trigonometric substitution.

a. $\int \frac{1}{x^2+16}\,dx$

b. $\int \frac{1}{\sqrt{25-x^2}}\,dx$

c. $\int \sqrt{36-x^2}\,dx$

d. Challenge problems (just for fun):

1. $\int x^3\sqrt{1-x^2}\,dx$
2. $\int x^5\sqrt{1-x^2}\,dx$

(a) Let $x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$$\int \frac{1}{x^2+16}\,dx = \int \frac{1}{16\tan^2\theta+16}\, 4\sec^2\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\tan^2\theta+1} \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\int d\theta = \frac{1}{4}\theta + C = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$$

(b) Let $x = 5\sin\theta$

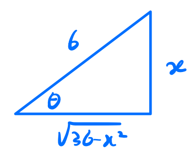
$dx = 5\cos\theta \, d\theta$

$$\int \frac{1}{\sqrt{25 - 25\sin^2\theta}} \quad 5\cos\theta \, d\theta$$

$$= \int d\theta = \theta + C = \arcsin\left(\frac{x}{5}\right) + C$$

(c) Let $x = 6\sin\theta$

$dx = 6\cos\theta \, d\theta$



$\sin\theta = \frac{x}{6}$

$\cos\theta = \frac{1}{6}\sqrt{36 - x^2}$

$$\int \sqrt{36 - 36\sin^2\theta} \quad 6\cos\theta \, d\theta$$

$$= \int 36\cos^2\theta \, d\theta$$

$$= 36\left(\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right) + C \quad \text{(using int. by parts, details not shown)}$$

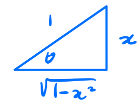
$$= 18\sin\theta\cos\theta + 18\theta + C$$

$$= 18\left(\frac{x}{6}\right)\left(\frac{\sqrt{36 - x^2}}{6}\right) + 18\arcsin\left(\frac{x}{6}\right) + C$$

$$= \frac{1}{2}x\sqrt{36 - x^2} + 18\arcsin\left(\frac{x}{6}\right) + C$$

8 Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$



$\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$$\int x^3 \sqrt{1-x^2}\, dx = \int \sin^3\theta \sqrt{1-\sin^2\theta}\; \cos\theta\, d\theta$$

$$= \int \sin^3\theta \cos^2\theta \, d\theta$$

$$= \int \sin\theta \cdot \sin^2\theta \cos^2\theta \, d\theta$$

$$= \int \sin\theta \,(1-\cos^2\theta)\cos^2\theta \, d\theta$$

$$= \int \sin\theta \cos^2\theta \, d\theta - \int \sin\theta \cos^4\theta \, d\theta$$

$w = \cos\theta$

$dw = -\sin\theta \, d\theta$

$-dw = \sin\theta \, d\theta$

$$= \int -w^2 \, dw + \int w^4 \, dw$$

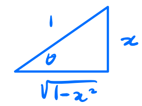
$$= -\frac{1}{3}w^3 + \frac{1}{5}w^5 + C$$

$$= -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C$$

$$= -\frac{1}{3}\left(1-x^2\right)^{3/2} + \frac{1}{5}\left(1-x^2\right)^{5/2} + C$$

(e) Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$



$\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$$\int x^5 \sqrt{1-x^2}\, dx = \int \sin^5\theta \sqrt{1-\sin^2\theta} \, \cos\theta \, d\theta$$

$$= \int \sin^5\theta \cos^2\theta \, d\theta$$

$$= \int \sin\theta \cdot \sin^4\theta \cos^2\theta \, d\theta$$

$$= \int \sin\theta \underbrace{(1-\cos^2\theta)^2}_{= 1 - 2\cos^2\theta + \cos^4\theta} \cos^2\theta \, d\theta$$

$$= \int \sin\theta \cos^2\theta \, d\theta - \int 2\sin\theta \cos^4\theta \, d\theta + \int \sin\theta \cos^6\theta \, d\theta$$

$w = \cos\theta$

$dw = -\sin\theta \, d\theta$

$-dw = \sin\theta \, d\theta$

$$= \int -w^2 \, dw + \int 2w^4 \, dw - \int w^6 \, dw$$

$$= -\frac{1}{3}w^3 + \frac{2}{5}w^5 - \frac{1}{7}w^7 + C$$

$$= -\frac{1}{3}\cos^3\theta + \frac{2}{5}\cos^5\theta - \frac{1}{7}\cos^7\theta + C$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{2}{5}(1-x^2)^{5/2} - \frac{1}{7}(1-x^2)^{7/2} + C.$$