

Ratio test Consider  $\sum_{n=1}^{\infty} a_n$  and let

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- If  $L < 1$ , then  $\sum a_n$  converges
- If  $L > 1$ , then  $\sum a_n$  diverges
- If  $L = 1$ , the test is inconclusive.

Example What does the ratio test tell us about the following series:

(1)  $\sum_{n=1}^{\infty} \frac{2^n}{n^3 + 1}$

(3)  $\sum_{n=1}^{\infty} \frac{(-8)^n}{(2n)!}$

(2)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

(4)  $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 5}$

$$\begin{aligned}
 \textcircled{1} \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)^3 + 1} \cdot \frac{n^3 + 1}{2^n} \\
 &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{2^n} \cdot \frac{n^3 + 1}{(n+1)^3 + 1} \\
 &= 2 \lim_{n \rightarrow \infty} \frac{n^3 + 1}{(n+1)^3 + 1} \\
 &= 2 > 1
 \end{aligned}$$

Therefore the series diverges by the ratio test

$$\begin{aligned}
 \textcircled{2} \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \\
 &= \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1}{(n+1)(n)(n-1)(n-2) \dots 3 \cdot 2 \cdot 1} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1
 \end{aligned}$$

Therefore the series converges by the ratio test

$$\begin{aligned}
(3) \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-8)^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(-8)^n} \right| \\
&= 8 \cdot \lim_{n \rightarrow \infty} \frac{(2n)!}{(2n+2)!} \\
&= 8 \cdot \lim_{n \rightarrow \infty} \frac{(2n)!}{(2n+2)(2n-1)(2n)!} \\
&= 8 \cdot \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n-1)} = 0 < 1
\end{aligned}$$

Therefore the series converges by the ratio test

$$\begin{aligned}
(4) \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+1)^3+5} \cdot \frac{n^3+5}{n^2} \\
&= 1 \quad (\text{using ratio of leading coeff's})
\end{aligned}$$

The ratio test is inconclusive. However,

we can use the limit comparison test to prove the series diverges, comparing it with  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

## Math 102 — Ratio test

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**Problem 1.** What does the ratio test tell us about each of the following series?

a.  $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

b.  $\sum_{n=1}^{\infty} \frac{4^n}{n!}$

c.  $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

d.  $\sum_{n=1}^{\infty} \frac{n!}{n^3}$

e.  $\sum_{n=1}^{\infty} \frac{n+2}{n^3+n}$

(a)  $a_n = \frac{1}{n3^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)3^{n+1}}}{\frac{1}{n3^n}}$$

$$= \lim_{n \rightarrow \infty} \frac{n3^n}{(n+1)3^{n+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}}$$

$$= \frac{1}{3} < 1, \text{ so } \sum_{n=1}^{\infty} \frac{1}{n3^n} \text{ converges}$$

(b)  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n}$

