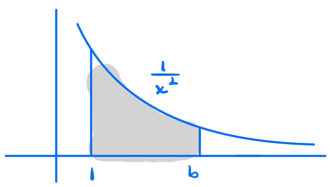
§ 7.6 Improper integrals

An improper integral is a definite integral where either

① one or both of the limits on integration is infinite

② the integrand $f(x)$ has a vertical asymptote at or between the limits of integration.

Example $\int_1^\infty \frac{1}{x^2}\,dx$



Let $b \to \infty$

We define $\int_1^\infty \frac{1}{x^2}\,dx$ to be $\lim_{b\to\infty}\left(\int_1^b \frac{1}{x^2}\,dx\right)$ and compute inside parentheses before taking the limit:

$$\lim_{b\to\infty}\left(\int_1^b \frac{1}{x^2}\,dx\right) = \lim_{b\to\infty}\left(\int_1^b x^{-2}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-x^{-1}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(1-b^{-1}\right)$$

$$= \lim_{b\to\infty}\left(1-\frac{1}{b}\right)$$

$= 1$. We say $\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1.

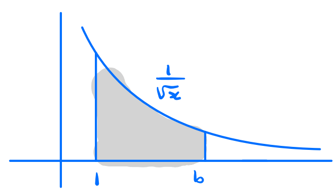
<u>Definition</u> Suppose $f(x) \geq 0$ for all $x \geq a$. We say

$\int_a^\infty f(x)\,dx$ <u>converges</u> if $\lim_{b\to\infty} \int_a^b f(x)\,dx < \infty$

Otherwise, we say $\int_a^\infty f(x)\,dx$ <u>diverges</u>.

<u>Example</u> $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$$\lim_{b\to\infty}\left(\int_1^b \frac{1}{\sqrt{x}}\,dx\right) = \lim_{b\to\infty}\left(\int_1^b x^{-1/2}\,dx\right)$$



$$= \lim_{b\to\infty}\left(2x^{1/2}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(2\left(b^{1/2}-1\right)\right)$$

Let $b\to\infty$

$$= \infty, \quad \text{so } \int_1^\infty \frac{1}{\sqrt{x}}\,dx \text{ diverges.}$$

<u>Example</u> $\int_0^\infty e^{-2x}\,dx$

$$\lim_{b\to\infty}\left(\int_0^b e^{-2x}\,dx\right) = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2x}\Big|_0^b\right)$$

$w = -2x$

$dw = -2dx$

$-\frac{1}{2}dw = dx$

$$= \lim_{b\to\infty} \frac{1}{2}\left(1-e^{-2b}\right)$$

$$= \frac{1}{2}\lim_{b\to\infty}\left(1-\frac{1}{e^{2b}}\right)$$

$$= \frac{1}{2}.$$

So $\int_0^\infty e^{-2x}\,dx = \frac{1}{2}$, or we say $\int_0^\infty e^{-2x}\,dx$ converges to $\frac{1}{2}$.

Problem 1. Determine whether the following improper integrals converge.

a. $\int_1^\infty \frac{1}{x^3}\,dx$

b. $\int_1^\infty \frac{1}{x^4}\,dx$

c. $\int_1^\infty \frac{1}{x^{1/3}}\,dx$

d. $\int_1^\infty \frac{1}{x^{4/5}}\,dx$

e. $\int_1^\infty \frac{1}{x}\,dx$

(a)

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-3}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2}x^{-2}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(\frac{1}{2}\left(1-\frac{1}{b^2}\right)\right)$$

$$= \frac{1}{2}$$ converges

(b)

$$\int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-4}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{3}x^{-3}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(\frac{1}{3}\left(1-\frac{1}{b^3}\right)\right)$$

$$= \frac{1}{3}$$ converges

(c)

$$\int_1^\infty \frac{1}{x^{1/3}}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-1/3}\,dx\right)$$

$$= \lim_{b\to\infty}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(\frac{3}{2}\left(b^{3/2}-1\right)\right)$$

$$= \infty$$ diverges

(d) $$\int_1^\infty \frac{1}{x^{4/5}}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-4/5}\,dx\right)$$

$$= \lim_{b\to\infty}\left(5x^{1/5}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(5\left(b^{1/5}-1\right)\right)$$

$$= \infty \quad \text{diverges}$$

(e) $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(\ln|x|\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(\ln b\right)$$

$$= \infty \quad \text{diverges.}$$

Problem 3. Consider the following integrals. Notice they all have finite limits of integration. Why should we consider them improper integrals? Make a guess about how we might define them in terms of limits and determine whether each converges or diverges.

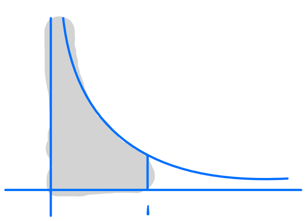
a. $\int_0^1 \frac{1}{x^{1/2}}\,dx$

b. $\int_0^1 \frac{1}{x^{1/3}}\,dx$

c. $\int_0^1 \frac{1}{x^2}\,dx$

d. $\int_0^1 \frac{1}{x^5}\,dx$

e. $\int_0^1 \frac{1}{x}\,dx$



(a) $$\int_0^1 \frac{1}{x^{1/2}}\,dx = \lim_{a\to 0^+}\left(\int_a^1 x^{-1/2}\,dx\right)$$

$$= \lim_{a\to 0^+}\left(2x^{1/2}\Big|_a^1\right)$$

$$= \lim_{a\to 0^+} 2\left(1-a^{1/2}\right)$$

$$= 2 \quad \text{converges}$$

(b) $$\int_0^1 \frac{1}{x^{1/3}}\,dx = \lim_{a\to 0^+}\left(\int_a^1 x^{-1/3}\,dx\right)$$

$$= \lim_{a\to 0^+}\left(\frac{3}{2}x^{2/3}\Big|_a^1\right)$$

$$= \lim_{a\to 0^+} \frac{3}{2}\left(1-a^{2/3}\right)$$

$$= \frac{3}{2}$$ converges

(c) $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left(\int_a^1 x^{-2}\,dx\right)$$

$$= \lim_{a\to 0^+}\left(-x^{-1}\Big|_a^1\right)$$

$$= \lim_{a\to 0^+} -\left(1-\frac{1}{a}\right)$$

$$= \infty$$ diverges

(d) $$\int_0^1 \frac{1}{x^5}\,dx = \lim_{a\to 0^+}\left(\int_a^1 x^{-5}\,dx\right)$$

$$= \lim_{a\to 0^+}\left(-\frac{1}{4}x^{-4}\Big|_a^1\right)$$

$$= \lim_{a\to 0^+} -\frac{1}{4}\left(1-\frac{1}{a^4}\right)$$

$$= \infty$$ diverges

(e) $$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left(\int_a^1 \frac{1}{x}\,dx\right)$$

$$= \lim_{a\to 0^+}\left(\ln|x|\Big|_a^1\right)$$

$$= \lim_{a\to 0^+}\left(-\ln a\right)$$

$$= \infty$$ diverges.