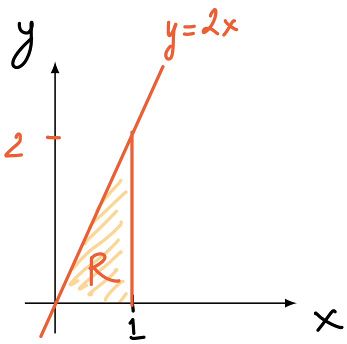
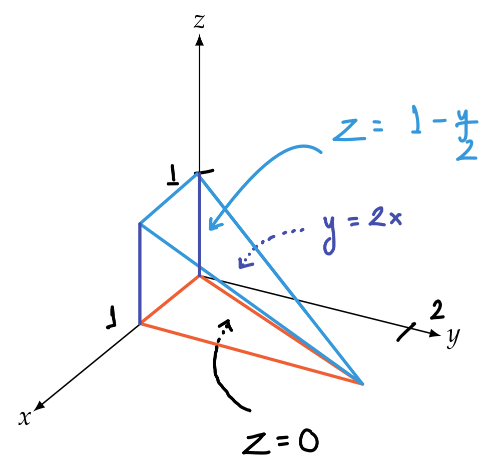
Let's practice setting up triple integrals using different orders of integration from $dV = dz\,dy\,dx$.

Example 1. Consider the iterated integral

$$\int_0^1 \int_0^{2x} \int_0^{1-\frac{y}{2}} f(x,y,z)\, \overset{\text{bounding surfaces}}{dz}\, \overbrace{dy\,dx}^{\text{shadow}}.$$

(a) Sketch the projection R of the region of integration D onto the xy-plane, and sketch D.

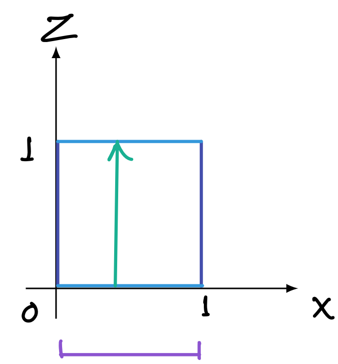




(b) Set up the triple integral using $dV = dy\,\overbrace{dz\,dx}^{\text{project onto } xz\text{-plane}}$.

bounding surface from $y = 0$ to $z = 1 - \frac{y}{2} \iff 2z = 2 - y \iff y = 2 - 2z$

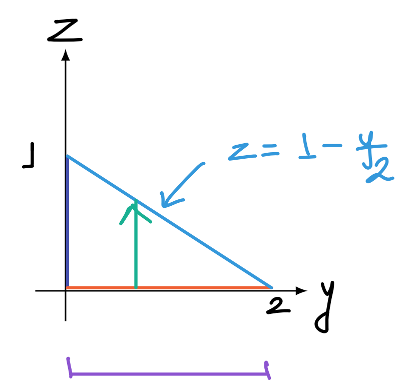
$$\int_{x=0}^{x=1} \int_{z=0}^{z=1} \int_{y=0}^{y=2-2z} f(x,y,z)\, dy\, dz\, dx$$



(c) Set up the triple integral using $dV = dx\,\underbrace{dz\,dy}_{\text{project onto } yz\text{-plane}}$.

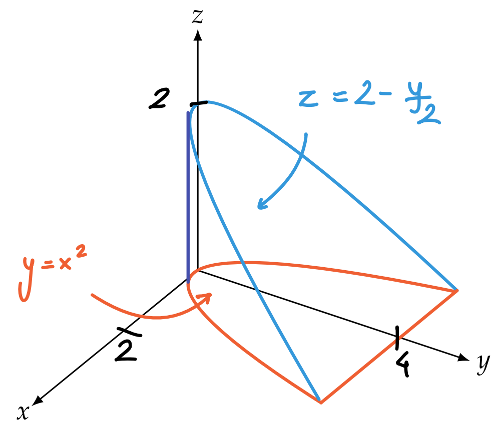
$$\int_{y=0}^{y=2} \int_{z=0}^{z=1-\frac{y}{2}} \int_{x=y/2}^{x=1} f(x,y,z)\, dx\, dz\, dy$$

bounding surfaces: from $y = 2x \iff x = y/2$ to $x = 1$



Example 2. Consider the solid region D bounded by the surfaces $y = x^2$, $2z = 4 - y$, and $z = 0$.

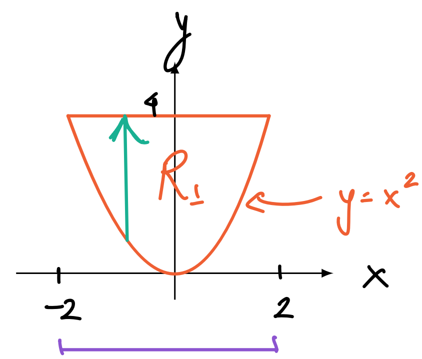
Draw and label the solid region D. Set up a triple integral for the volume of D using the following choices of dV.



project R_1 onto xy-plane

(a) $dV = dz\,\underbrace{dy\,dx}$

$$\int_{x=-2}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=2-\frac{y}{2}} 1\,dz\,dy\,dx$$

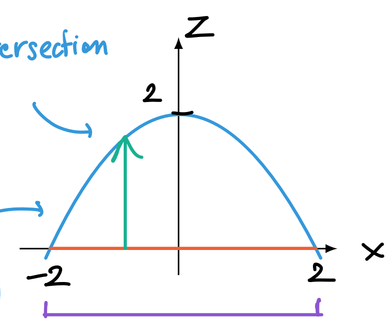


(b) $dV = dy\,\underbrace{dz\,dx}$ project onto xz-plane

$$\int_{x=-2}^{x=2}\int_{z=0}^{z=2-\frac{x^2}{2}}\int_{y=x^2}^{y=4-2z} 1\,dy\,dz\,dx$$

given by intersection of $z = 2 - \frac{y}{2}$ and $y = x^2$:

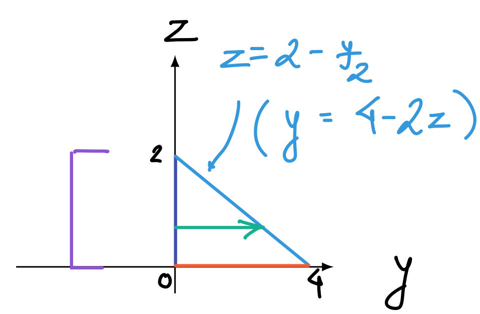
$$\boxed{z = 2 - \frac{x^2}{2}}$$



(c) $dV = dx\,\underbrace{dy\,dz}$ project onto yz-plane

$$\int_{z=0}^{z=2}\int_{y=0}^{y=4-2z}\int_{x=-\sqrt{y}}^{x=\sqrt{y}} 1\,dx\,dy\,dz$$

bounding surfaces.

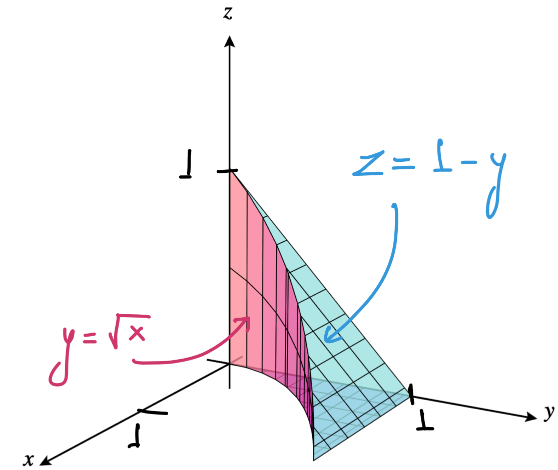


Example 3. The figure shows the region of integration D for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx.$$

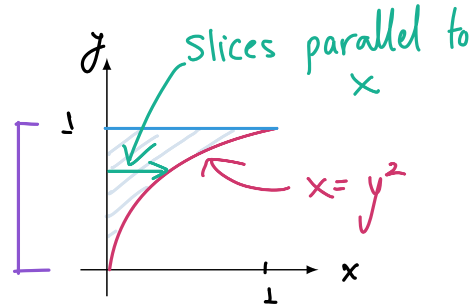
Label the surfaces that bound the solid region D. Rewrite this integral using:

$y = \sqrt{x}$ $\qquad$ $z = 1 - y$



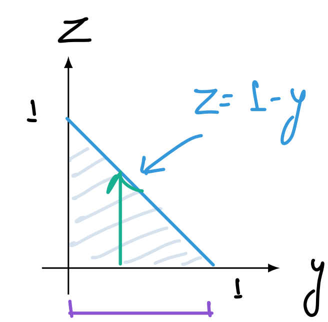
(a) $dV = \underbrace{dz\,dx\,dy}$ project onto xy-plane

$$\int_{y=0}^{y=1} \int_{x=0}^{x=y^2} \int_{z=0}^{z=1-y} f(x,y,z)\,dz\,dx\,dy$$



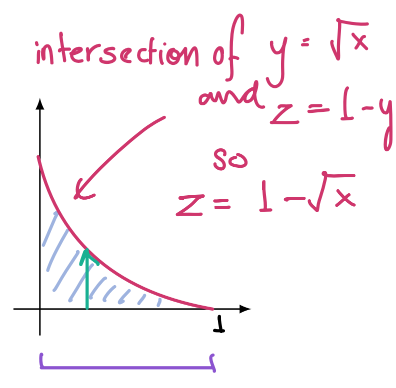
(b) $dV = \underbrace{dx\,dz\,dy}$ project onto yz-plane

$$\int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \int_{x=0}^{x=y^2} f(x,y,z)\,dx\,dz\,dy$$



(c) $dV = dy\,dz\,dx$ project onto xz-plane

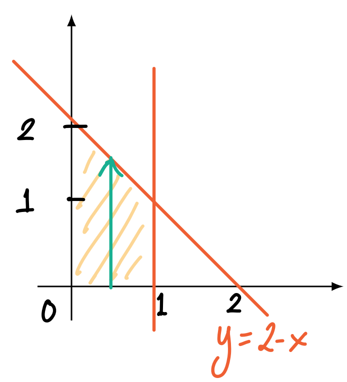
$$\int_{x=0}^{x=1} \int_{z=0}^{z=1-\sqrt{x}} \int_{y=\sqrt{x}}^{y=1-z} f(x,y,z)\,dy\,dz\,dx$$

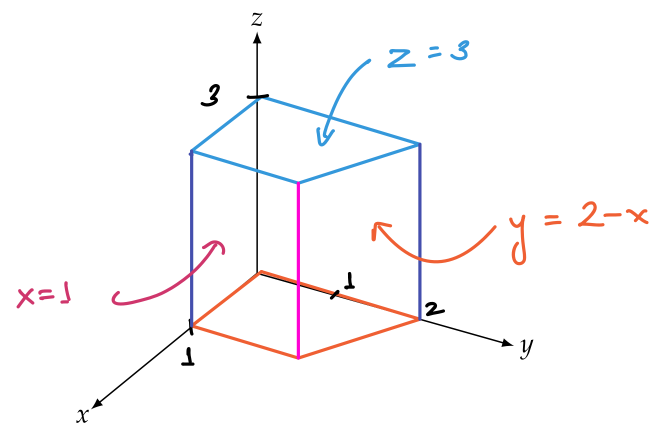


Example 4. Consider the iterated integral

$$\int_0^1 \int_0^{2-x} \int_0^3 f(x,y,z)\,dz\,dy\,dx.$$

(a) Sketch the projection R of the region of integration D onto the appropriate coordinate plane, and sketch D.

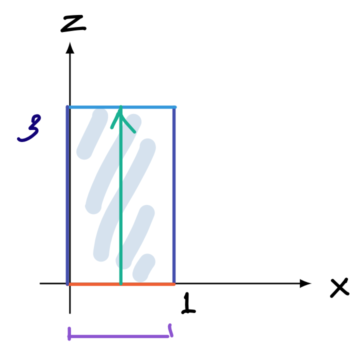




(b) Set up the triple integral using $dV = dy\,\underbrace{dz\,dx}$.

project onto xz-plane

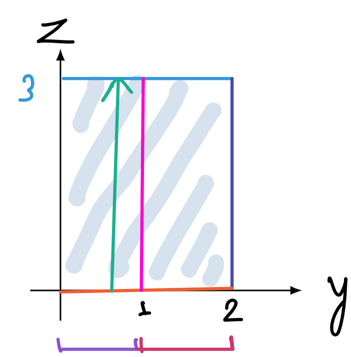
$$\int_{x=0}^{x=1} \int_{z=0}^{z=3} \int_{y=0}^{y=2-x} f(x,y,z)\,dy\,dz\,dx$$



(c) Set up the triple integral using $dV = dx\,dz\,dy$.

$$\int_{y=0}^{y=1} \int_{z=0}^{z=3} \int_{x=0}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

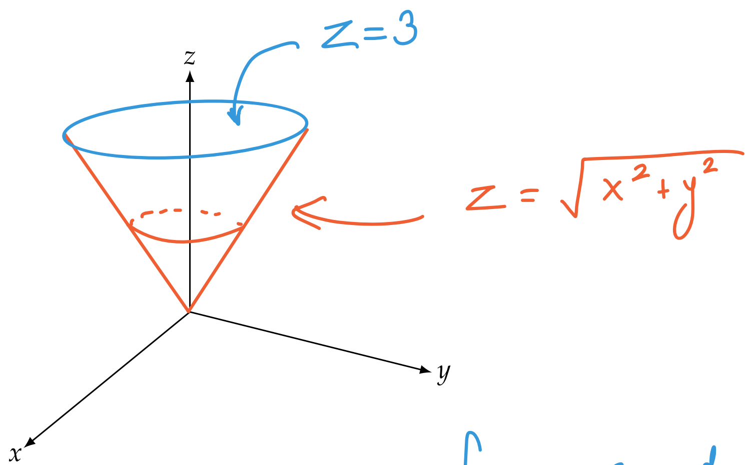
$$+ \int_{y=1}^{y=2} \int_{z=0}^{z=3} \int_{x=0}^{x=2-y} f(x,y,z)\,dx\,dz\,dy$$



More Practice with Triple Integrals

Example 5. Let D be the solid region bounded between the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 3$.

(a) Sketch and label D.



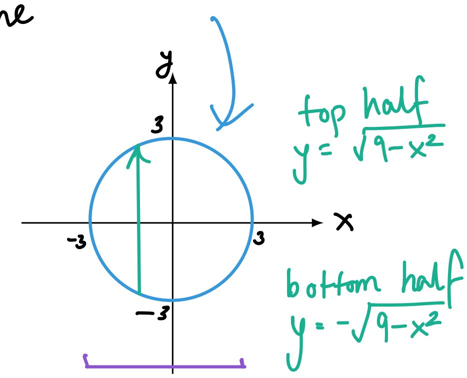
intersection of $Z = 3$ and $Z = \sqrt{x^2+y^2}$

$3 = \sqrt{x^2+y^2}$

$9 = x^2 + y^2$ (circle of radius 3)

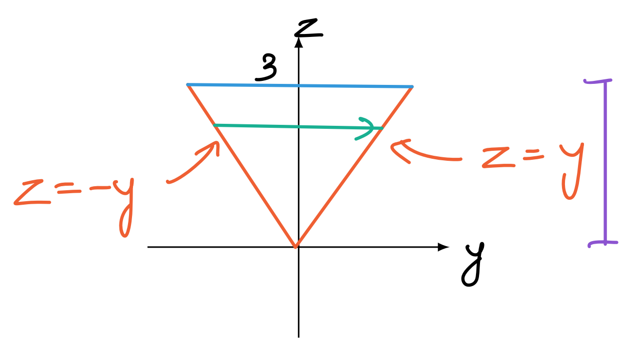
(b) Set up the triple integral using $dV = dz\, dy\, dx$. ← project onto xy-plane

$$\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=3} f(x,y,z)\, dz\, dy\, dx$$



(c) Set up the triple integral using $dV = dx\, dy\, dz$.

$$\int_{z=0}^{z=3} \int_{y=-z}^{y=z} \int_{x=-\sqrt{z^2-y^2}}^{x=\sqrt{z^2-y^2}} f(x,y,z)\, dx\, dy\, dz$$



bounding surfaces from solving

$z = \sqrt{x^2+y^2}$

$z^2 = x^2 + y^2$

$x = \pm\sqrt{z^2 - y^2}$.

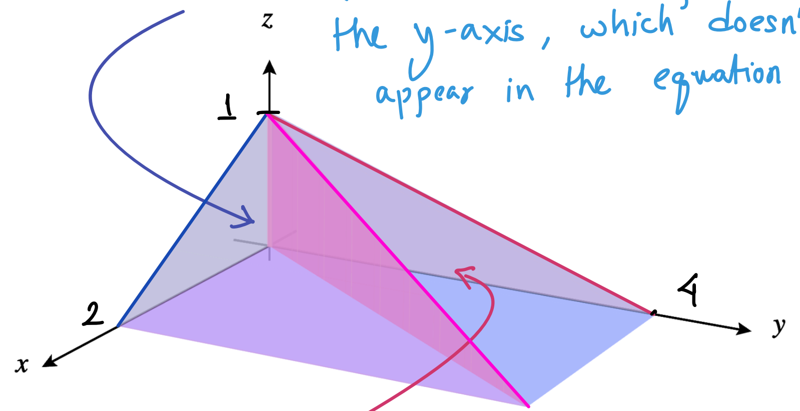
$\left(z = 1 - \frac{1}{2}x\right)$

$x = 2 - 2z$ (notice it runs parallel to the y-axis, which doesn't appear in the equation $x = 2-2z$)

Example 6. The figure shows the region of integration D—a pyramid-shaped solid with a rectangular base—for the integral

$$\int_0^4 \int_0^{1-\frac{y}{4}} \int_0^{2-2z} f(x,y,z)\,dx\,dz\,dy.$$

Label the surfaces that bound the D. Rewrite this integral using:



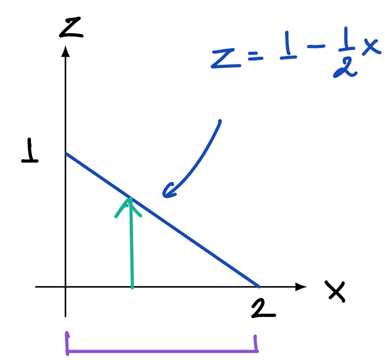
$z = 1 - \frac{y}{4}$

(notice it runs parallel to the x-axis)

$(y = 4 - 4z)$

(a) $dV = dy\,\underbrace{dz\,dx}$ project onto xz-plane

$$\int_{x=0}^{x=2} \int_{z=0}^{z=1-\frac{1}{2}x} \int_{y=0}^{y=4-4z} f(x,y,z)\,dy\,dz\,dx$$



(b) $dV = dz\,dy\,dx$

$$\int_{x=0}^{x=2} \int_{y=2x}^{y=4} \int_{z=0}^{z=1-\frac{y}{4}} f(x,y,z)\,dz\,dy\,dx$$

$$+ \int_{x=0}^{x=2} \int_{y=0}^{y=2x} \int_{z=0}^{z=1-\frac{1}{2}x} f(x,y,z)\,dz\,dy\,dx$$

$y = 2x$