12.3 Partial Derivatives

Def Let $f(x,y)$ be a given function.

The partial derivative of f with respect to x is

treat y like a constant $\longrightarrow$ $$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \left(\text{denoted } f_x \text{ or } \frac{\partial f}{\partial x}\right)$$

Similarly, the partial derivative with respect to y is

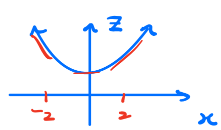
treat x like a constant $\longrightarrow$ $$\lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \quad \left(\text{denoted } f_y \text{ or } \frac{\partial f}{\partial y}\right)$$

curly ∂ is used for multivariable function partial derivative

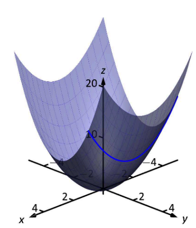
Example Let $f(x,y) = x^2+2y^2$

(1) Compute f_x, $f_x(-2,2)$, $f_x(0,2)$, $f_x(2,2)$ and discuss what this means geometrically about slopes along the curve $z = f(x,2)$.

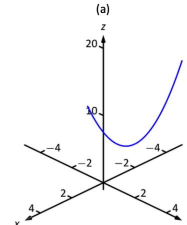
$f_x = 2x$



$z = f(x,2)$
$= x^2+8$



(a)



(b)

$f_x(-2,2) = -4 \Rightarrow$ negative slope at $(-2,2)$

$f_x(0,2) = 0 \Rightarrow 0$ slope at $(0,2)$

$f_x(2,2) = 4 \Rightarrow$ positive slope at $(2,2)$

slopes along curve where $y=2$

② Do the same with f_y, $f_y(1,-2)$, $f_y(1,0)$, $f_y(1,2)$

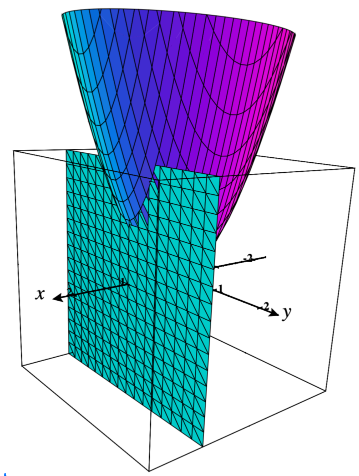
$f_y = 4y$

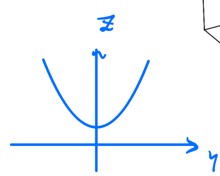
$f_y(1,-2) = -8 \Rightarrow$ negative slope at $(1,-2)$

$f_y(1,0) = 0 \Rightarrow 0$ slope at $(1,0)$

$f_y(1,2) = 8 \Rightarrow$ positive slope at $(1,2)$

$\underbrace{\phantom{\text{positive slope at}}}_{}$ slopes along curve where $x=1$





$z = 1+2y^2$

General Intuition

① $$f_x(a,b) \approx \frac{\Delta z}{\Delta x} \quad \text{at } (a,b) \text{ with } y \text{ fixed constant}$$

$f_x(a,b) > 0 \Rightarrow \Delta z > 0$ when x increases from a and y held constant at b

$f_x(a,b) < 0 \Rightarrow \Delta z < 0$ when x increases from a and y held constant at b

① $$f_y(a,b) \approx \frac{\Delta z}{\Delta y} \quad \text{at } (a,b) \text{ with } x \text{ fixed constant}$$

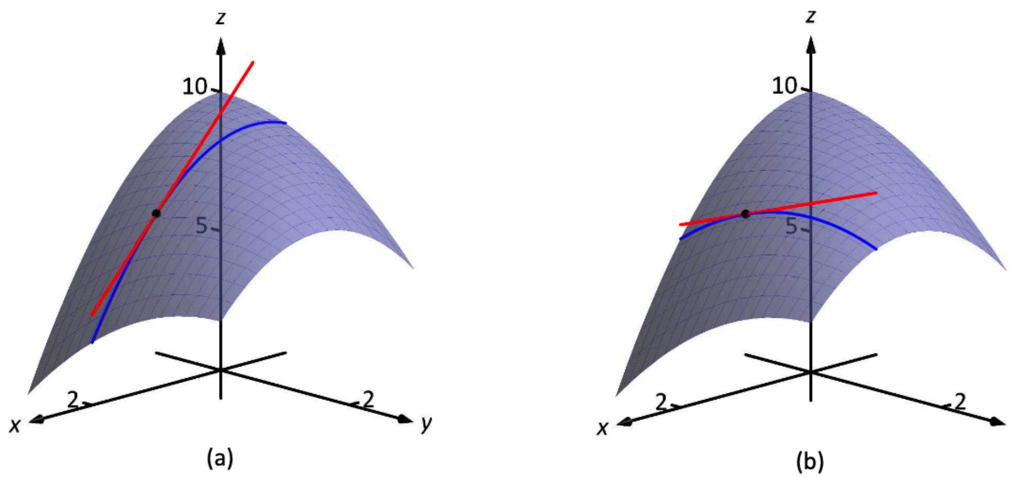
$f_y(a,b) > 0 \Rightarrow \Delta z > 0$ when y increases from b and x held constant at a

$f_y(a,b) < 0 \Rightarrow \Delta z < 0$ when y increases from b and x held constant at a

Example Let $f(x,y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2,1)$ and $f_y(2,1)$.

Which is bigger: $f(2,1)$ or $f(2.1,1)$?

$f(2,1)$ or $f(2,1.1)$?



$f_x = -2x + y$,

$f_x(2,1) = -2(2) + 1 = -3 < 0$

(negative slope)

$f_y = -y + x$,

$f_y(2,1) = -1 + 2 = 1 > 0$

(positive slope)

Problem 1. Compute f_x and f_y for each function below.

a. $f(x,y) = 5x^2y^3 + 8xy^2 - 3x^2$

b. $f(x,y) = \sin(x^2y^3) + \cos(y^2)$

c. $f(x,y) = xe^{x^2y^2}$

(a) $f_x = 10xy^3 + 8y^2 - 6x$

$f_y = 15x^2y^2 + 16xy$

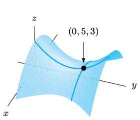
(b) $f_x = 2xy^3\cos(x^2y^3)$

$f_y = 3x^2y^2\cos(x^2y^3) - 2y\sin(y^2)$

(c) $f_x = e^{x^2y^2} + 2x^2y^2e^{x^2y^2}$ (using product rule)

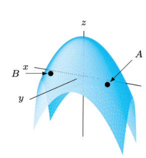
$f_y = 2x^3y\,e^{x^2y^2}$

Problem 2. The figure below shows the graph of $f(x,y)$. What are the signs of $f_x(0,5)$ and $f_y(0,5)$?



$f_x(0,5)$ negative, $f_y(0,5)$ positive

Problem 3. The figure below shows the graph of $f(x,y)$. What are the signs of $f_x(A)$, $f_y(A)$, $f_x(B)$, and $f_y(B)$?



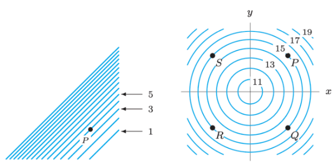
$f_x(A)$ positive

$f_x(B)$ negative

$f_y(A)$ negative

$f_y(B)$ negative

Problem 4. The figures below show the contour plots of the functions $g(x, y)$ and $f(x, y)$. Notice the curves are labeled with corresponding z values. What are the signs of $g_x(P)$ and $g_y(P)$ and of f_x and f_y at the points P, Q, R, and S? Positive, negative, or zero?



$g_x(P) < 0, \quad g_y(P) > 0$

$f_x(P) > 0 \quad f_x(Q) > 0 \quad f_x(R) < 0 \quad f_x(S) < 0$

$f_y(P) > 0 \quad f_y(Q) < 0 \quad f_y(R) < 0 \quad f_y(S) > 0$