

12.3 Second order partial derivatives

Last time we introduced f_x and f_y as partial derivatives of f with respect to each variable x and y .

Def The second order partial derivatives of f are the partial derivatives of f_x and f_y :

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \quad (\text{derivative of } f_x \text{ wrt } x)$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} \quad (\text{derivative of } f_y \text{ wrt } y)$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} \quad (\text{derivative of } f_x \text{ wrt } y)$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} \quad (\text{derivative of } f_y \text{ wrt } x)$$

Example Compute f_{xx} , f_{yy} , f_{xy} , f_{yx} when

① $f(x,y) = x^2 y^3$

② $f(x,y) = y \cos x$

① $f_x = 2xy^3$

$f_y = 3x^2 y^2$

$f_{xx} = 2y^3$

$f_{yy} = 6x^2 y$

$f_{xy} = 6xy^2$

$f_{yx} = 6xy^2$

② $f_x = -y \sin x$

$f_y = \cos x$

$f_{xx} = -y \cos x$

$f_{yy} = 0$

$f_{xy} = -\sin x$

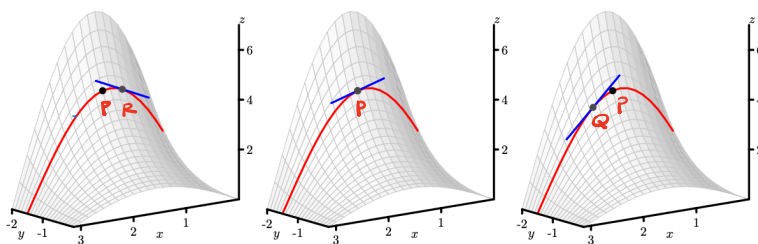
$f_{yx} = -\sin x$

Theorem (Clairaut's Theorem) When f_{xy} and f_{yx} are continuous, $f_{yx} = f_{xy}$

Rule of Thumb $f_{yx} = f_{xy}$ for the functions we'll encounter this semester.

Geometric meaning of second order partials

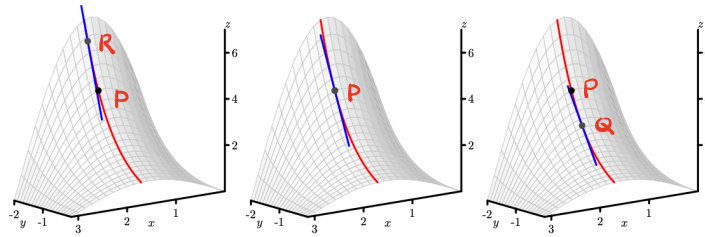
Question What can be said about slopes and first order partials in the image below?



$$f_x(R) > 0 \quad f_x(P) < 0 \quad f_x(Q) < 0$$

f_x changes from positive to negative (decreases)
as x increases and y stays constant

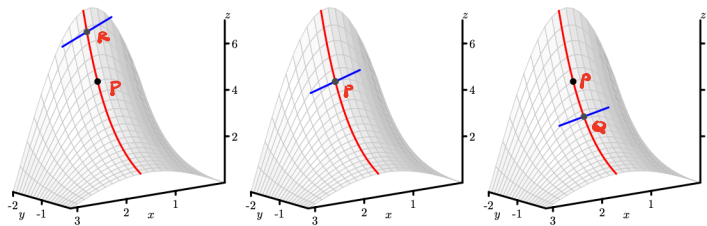
$$\Rightarrow f_{xx}(P) < 0 \quad \cap \quad \text{concave down}$$



$$f_y(R) < 0 \quad f_y(P) < 0 \quad f_y(Q) < 0$$

f_y changes from very negative to less negative (increases)
as y increases and x stays constant

$$\Rightarrow f_{yy}(P) > 0 \quad \text{concave up}$$

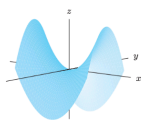


$$f_x(R) < 0 \quad f_x(P) < 0 \quad f_x(Q) < 0$$

f_x changes from more negative to less negative (increases)
as y increases and x stays constant

$$\Rightarrow f_{xy}(P) > 0 \quad (\text{not exactly related to concavity})$$

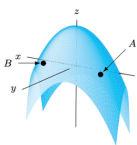
Problem 1. The figure below shows the graph of $f(x, y)$. What are the signs of $f_x(0, 0)$, $f_y(0, 0)$, $f_{xx}(0, 0)$, and $f_{yy}(0, 0)$?



$$f_x(0, 0) = 0 \quad f_{xx}(0, 0) > 0 \quad \text{++}$$

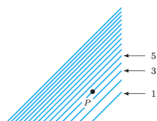
$$f_y(0, 0) = 0 \quad f_{yy}(0, 0) < 0 \quad \text{--}$$

Problem 2. Let P be a point on the line segment connecting A to B in the figure below. How does $f_x(P)$ change when P starts close A and moves toward B ? What can you conclude about any second order partial derivatives as a result?



$f_x(A) > 0$ but $f_x(B) < 0$, so $f_x(P)$ decreases as P goes from A to B , which implies $f_{xx}(P) < 0$

Problem 3. The figure below shows a contour plot of $f(x, y)$. What are the signs of $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{yy}(P)$, and $f_{xy}(P)$?



$$f_x(P) < 0, \quad f_{xx}(P) > 0, \quad f_{xy}(P) < 0$$

$$f_y(P) > 0, \quad f_{yy}(P) > 0$$

Problem 4. Compute all four second order partial derivatives for each f below and check that $f_{xy} = f_{yx}$.

- a. $f(x, y) = 3x^2y + 5xy^3$
- b. $f(x, y) = e^{2xy}$
- c. $f(x, y) = \sin(x^2 + y^2)$

Ⓐ

$$f_x = 6xy + 5y^3$$

$$f_{xx} = 6y$$

$$f_{xy} = 6x + 15y^2$$

$$f_y = 3x^2 + 15xy^2$$

$$f_{yy} = 30xy$$

$$f_{yx} = 6x + 15y^2$$

Ⓑ

$$f_x = 2ye^{2xy}$$

$$f_{xx} = 4y^2e^{2xy}$$

$$f_{xy} = 2e^{2xy} + 4xye^{2xy}$$

$$f_y = 2xe^{2xy}$$

$$f_{yy} = 4x^2e^{2xy}$$

$$f_{yx} = 2e^{2xy} + 4xye^{2xy}$$

Ⓒ

$$f_x = 2x \cos(x^2 + y^2)$$

$$f_{xx} = 2 \cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2)$$

$$f_{xy} = -4xy \sin(x^2 + y^2)$$

$$f_y = 2y \cos(x^2 + y^2)$$

$$f_{yy} = 2 \cos(x^2 + y^2) - 4y^2 \sin(x^2 + y^2)$$

$$f_{yx} = -4xy \sin(x^2 + y^2)$$