12.3 Second order partial derivatives

Last time we introduced f_x and f_y as partial derivatives of f with respect to each variable x and y.

Def The second order partial derivatives of f are the partial derivatives of f_x and f_y:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \quad \text{(derivative of } f_x \text{ wrt } x\text{)}$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} \quad \text{(derivative of } f_y \text{ wrt } y\text{)}$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} \quad \text{(derivative of } f_x \text{ wrt } y\text{)}$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} \quad \text{(derivative of } f_y \text{ wrt } x\text{)}$$

Examples Compute f_{xx}, f_{yy}, f_{xy}, f_{yx} when

① $f(x,y) = x^2y^3$

② $f(x,y) = y\cos x$

① $f_x = 2xy^3$ $\quad f_y = 3x^2y^2$

$f_{xx} = 2y^3$ $\quad f_{yy} = 6x^2y$

$f_{xy} = 6xy^2$ $\quad f_{yx} = 6xy^2$

② $f_x = -y\sin x$ $\quad f_y = \cos x$

$f_{xx} = -y\cos x$ $\quad f_{yy} = 0$

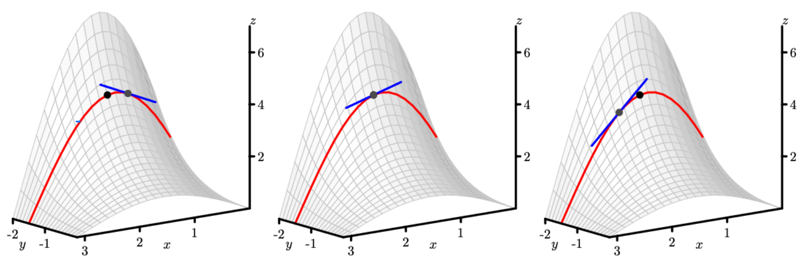
$f_{xy} = -\sin x$ $\quad f_{yx} = -\sin x$

Theorem (Clairaut's Theorem) When f_{xy} and f_{yx} are continuous, $f_{yx} = f_{xy}$

Rule of Thumb $f_{xy} = f_{yx}$ for the functions we'll encounter this semester.

Geometric meaning of second order partials

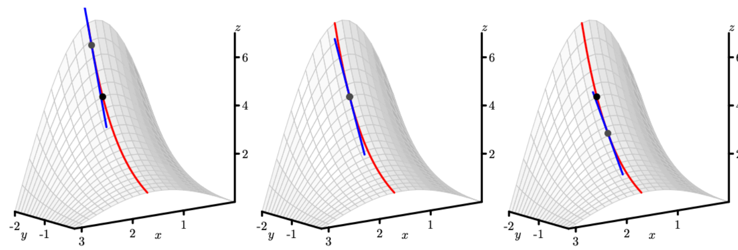
Question What can be said about slopes and first order partials in the image below?



$f_x > 0$ $\qquad$ $f_x < 0$ $\qquad$ $f_x < 0$

f_x changes from positive to negative (decreases) as x increases and y stays constant

$\Rightarrow f_{xx} < 0$ $\frown$ concave down

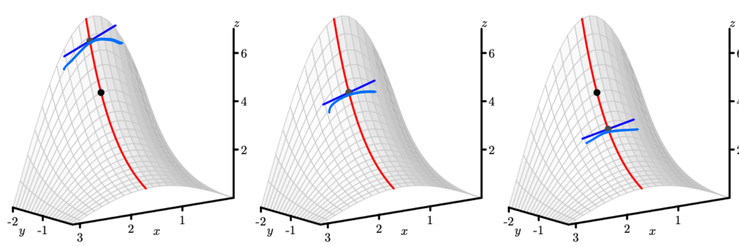


$f_y < 0$ $\qquad$ $f_y < 0$ $\qquad$ $f_y < 0$

f_y changes from very negative to less negative (increases)

as y increases and x stays constant

$\Rightarrow$ $f_{yy} > 0$ $\quad$ concave up



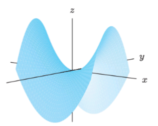
$f_x < 0$ $\qquad$ $f_x < 0$ $\qquad$ $f_x < 0$

f_x changes from more negative to less negative (increases)

as y increases and x stays constant

$\Rightarrow$ $f_{xy} > 0$ (not exactly related to concavity)

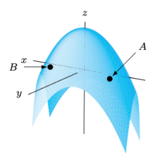
Problem 1. The figure below shows the graph of $f(x, y)$. What are the signs of $f_x(0, 0)$, $f_y(0, 0)$, $f_{xx}(0, 0)$, and $f_{yy}(0, 0)$?



$f_x(0,0) = 0$ $\qquad$ $f_{xx}(0,0) > 0$

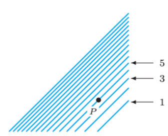
$f_y(0,0) = 0$ $\qquad$ $f_{yy}(0,0) < 0$

Problem 2. Let P be a point on the line segment connecting A to B in the figure below. How does $f_x(P)$ change when P starts close A and moves toward B? What can you conclude about any second order partial derivatives as a result?



$f_x(A) > 0$ but $f_x(B) < 0$, so $f_x(P)$ decreases as P goes from A to B, which implies $f_{xx}(P) < 0$

Problem 3. The figure below shows a contour plot of $f(x, y)$. What are the signs of $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{yy}(P)$, and $f_{xy}(P)$?



$f_x(P) < 0$, $\quad f_{xx}(P) > 0$ $\qquad f_{xy}(P) < 0$

$f_y(P) > 0$, $\quad f_{yy}(P) > 0$

Problem 4. Compute all four second order partial derivatives for each f below and check that $f_{xy} = f_{yx}$.

a. $f(x, y) = 3x^2y + 5xy^3$

b. $f(x, y) = e^{2xy}$

c. $f(x, y) = \sin(x^2 + y^2)$

(a) $f_x = 6xy + 5y^3$

$f_{xx} = 6y$

$f_{xy} = 6x + 15y^2$

$f_y = 3x^2 + 15xy^2$

$f_{yy} = 30xy$

$f_{yx} = 6x + 15y^2$

(b) $f_x = 2ye^{2xy}$

$f_{xx} = 4y^2e^{2xy}$

$f_{xy} = 2e^{2xy} + 4xye^{2xy}$

$f_y = 2xe^{2xy}$

$f_{yy} = 4x^2e^{2xy}$

$f_{yx} = 2e^{2xy} + 4xye^{2xy}$

(c) $f_x = 2x\cos(x^2+y^2)$

$f_{xx} = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$

$f_{xy} = -4xy\sin(x^2+y^2)$

$f_y = 2y\cos(x^2+y^2)$

$f_{yy} = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$

$f_{yx} = -4xy\sin(x^2+y^2)$