

## 12.6 Directional Derivatives

Goal We've learned how partial derivatives  $f_x$  and  $f_y$  tells us about how  $f$  changes as we move in  $x$  or  $y$  directions. We'd like to define and study derivatives (i.e. rates of change) of  $f(x,y)$  in any arbitrary direction given by a unit vector  $\vec{u} \in \mathbb{R}^2$ .

Def Let  $\vec{u} = \langle u_1, u_2 \rangle$  be a unit vector and  $P = (a,b) \in \mathbb{R}^2$ .

The directional derivative of  $f$  at  $P$  in the direction of  $\vec{u}$  is given by

$$D_{\vec{u}} f(P) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

• if  $\vec{v}$  is not a unit vector we define

$$D_{\vec{v}} f(P) = D_{\vec{u}} f(P) \text{ where } \vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}$$

(we only care about direction, not length of  $\vec{v}$ )

• if  $\vec{u} = \vec{i} = \langle 1, 0 \rangle$ ,  $D_{\vec{u}} f(P) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a,b)}{h} = f_x(P)$

• if  $\vec{u} = \vec{j} = \langle 0, 1 \rangle$ ,  $D_{\vec{u}} f(P) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a,b)}{h} = f_y(P)$

How do we compute  $D_{\vec{u}}f(P)$  without limits?

Remember the local linear approximation of  $f$  near  $(a,b)$ :

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Then

*this becomes an equality as  $h \rightarrow 0$*

$$\begin{aligned} f(a+hu_1, b+hu_2) &\approx L(a+hu_1, b+hu_2) \\ &= f(a,b) + f_x(a,b)hu_1 + f_y(a,b)hu_2 \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} &\approx \frac{f_x(a,b)hu_1 + f_y(a,b)hu_2}{h} \\ &= f_x(a,b)u_1 + f_y(a,b)u_2 \end{aligned}$$

Thus

$D_{\vec{u}}f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \langle f_x(a,b), f_y(a,b) \rangle \cdot \vec{u}$

*gradient vector*

Example Find the directional derivative of

$$f(x,y) = y^3 - 3xy + x^2 + 3$$

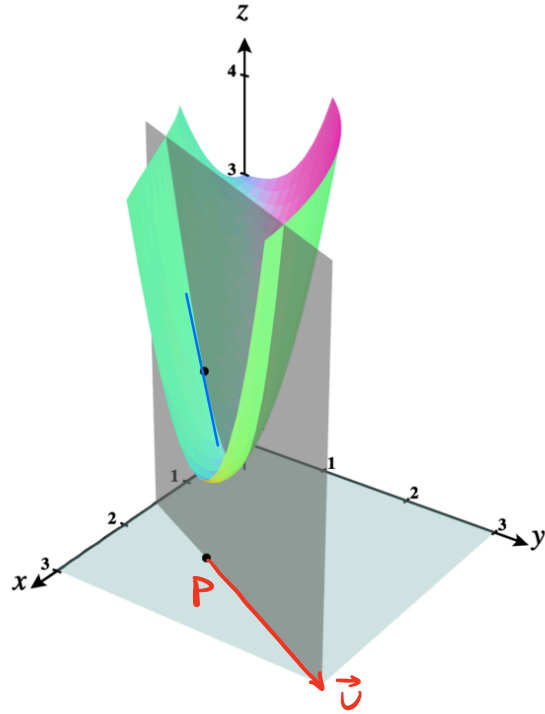
at  $P = (2,1)$  in the direction of  $\vec{v} = \langle 1, 2 \rangle$ .

$$\begin{aligned} f_x &= -3y + 2x, & f_x(P) &= -3 + 4 = 1 & \vec{u} &= \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle \\ f_y &= 3y^2 - 3x, & f_y(P) &= 3 - 6 = -3 \end{aligned}$$

$$D_{\vec{v}}f(P) = \langle f_x(P), f_y(P) \rangle \cdot \vec{u} = \langle 1, -3 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{-5}{\sqrt{5}} = -\sqrt{5}$$

Geometrically...

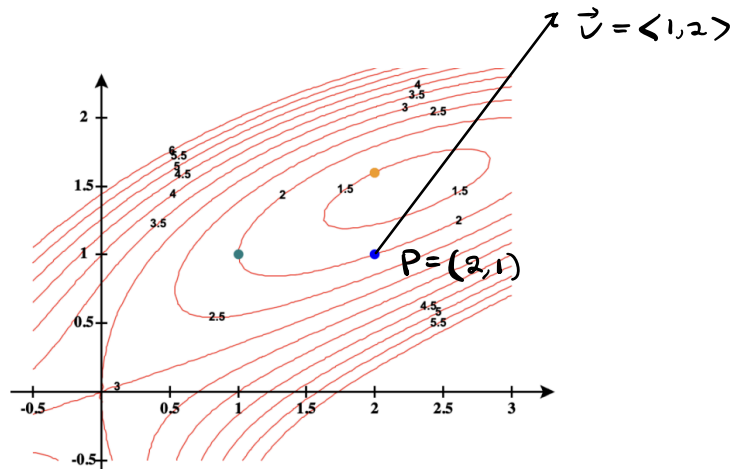
$D_{\vec{v}}f(2,1)$  is  
the slope of the  
tangent line to the  
curve at the intersection  
of the graph of  $f(x,y)$   
and the vertical plane  
through  $(2,1,f(2,1))$  parallel  
to  $\vec{v}$



It helps us understand

$\Delta f$  as we move away  
from  $(2,1)$  in the direction of  $\vec{v}$

The contour plot of  $f \dots$



notice  $D_{\vec{v}} f(2,1) < 0$  since it only describes how the function values change near  $(2,1)$  in the direction of  $\vec{v}$  (not how  $f$  changes far away from  $(2,1)$ )

Let  $\vec{w} = \langle -\frac{1}{2}, -1 \rangle$ . Determine the signs of

$$D_{\vec{w}} f(1,1) \quad \text{and} \quad D_{\vec{w}} f(2,1)$$

$$\underbrace{\hspace{1.5cm}}_{+} \quad \underbrace{\hspace{1.5cm}}_{+}$$

Which is bigger?

$D_{\vec{w}} f(2,1)$  since the level curves are more tightly spaced near  $(2,1)$ , so the  $f$  values change more quickly there.

Def The gradient vector of  $f$  at  $(a,b)$  is

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

The gradient of  $f$  is the vector-valued function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

or just  $\nabla f = \langle f_x, f_y \rangle$ .

Example Let  $\vec{w} = \langle -\frac{1}{2}, -1 \rangle$  and  $f(x,y) = y^3 - 3xy + x^2 + 3$

Compute  $\nabla f$ ,  $\nabla f(1,1)$ , and  $D_{\vec{w}} f(1,1)$ .

$$\nabla f = \langle -3y + 2x, 3y^2 - 3x \rangle$$

$$\nabla f(1,1) = \langle -1, 0 \rangle$$

$$\begin{aligned} D_{\vec{w}} f(1,1) &= \nabla f(1,1) \cdot \frac{1}{\|\vec{w}\|} \vec{w} \\ &= \langle -1, 0 \rangle \cdot \left( \frac{1}{\sqrt{5/4}} \right) \langle -\frac{1}{2}, -1 \rangle \\ &= \frac{1}{\sqrt{5}} \end{aligned}$$

Why is the gradient vector important when thinking about the directional derivative?

$$\begin{aligned}\text{Notice } D_{\vec{u}} f(a,b) &= \nabla f(a,b) \cdot \vec{u} \\ &= \|\nabla f(a,b)\| \|\vec{u}\| \cos \theta \\ &= \|\nabla f(a,b)\| \cos \theta\end{aligned}$$

Unit vector

where  $\theta$  is the angle between  $\nabla f(a,b)$  and  $\vec{u}$

Suppose we let  $\vec{u}$  vary (ie. we're allowed to pick the direction for our directional derivative):

① What choice of  $\vec{u}$  maximizes  $D_{\vec{u}} f(a,b)$ ?

$D_{\vec{u}} f(a,b)$  is maximized when  $\cos \theta = 1$ , ie  $\theta = 0$ ,  
ie.  $\vec{u}$  points in direction of  $\nabla f(a,b)$

The maximum rate of increase is  $\|\nabla f(a,b)\|$

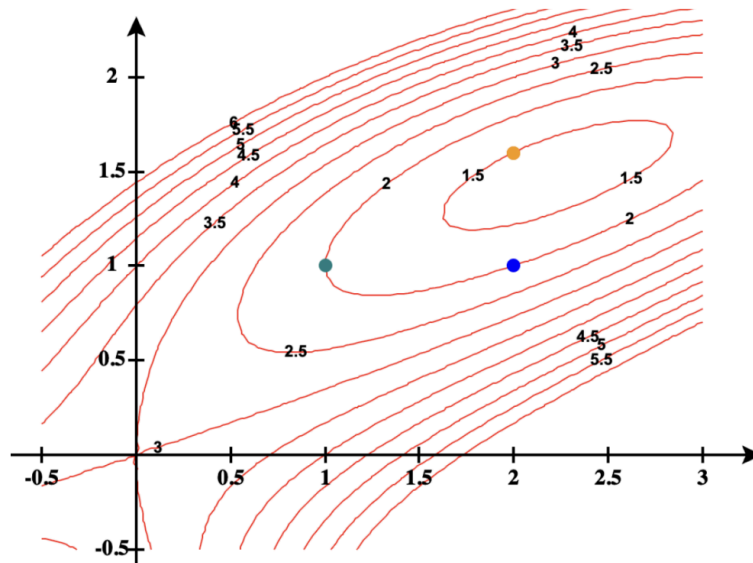
② What choice of  $\vec{u}$  minimizes  $D_{\vec{u}} f(a,b)$ ?

$D_{\vec{u}} f(a,b)$  is minimized when  $\cos \theta = -1$ , ie  $\theta = \pi$ ,  
ie.  $\vec{u}$  points in direction of  $-\nabla f(a,b)$

The maximum rate of decrease is  $-\|\nabla f(a,b)\|$

③  $D_{\vec{u}} f(a,b) = 0$  when  $\cos \theta = 0$ , when  $\theta = \frac{\pi}{2}$ ,

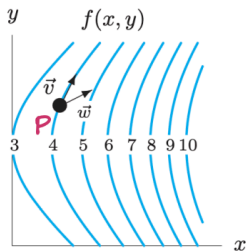
ie.  $\vec{u}$  is orthogonal to  $\nabla f(a,b)$



- Given that  $\nabla f(2,1)$  points in direction of maximum possible increase at  $(2,1)$  which way should it point?
- Which way takes us to the next larger contour quickest?
- We found  $\nabla f(1,1) = \langle -1, 0 \rangle$ . Is that reasonable?
- Sketch  $\nabla f(2,1)$  and  $\nabla f(2,1.6)$  above. Which is longer?
- If  $\vec{w}$  points tangent to the level curve at  $(1,1)$ , what is the value of  $D_{\vec{w}} f(1,1)$ ?

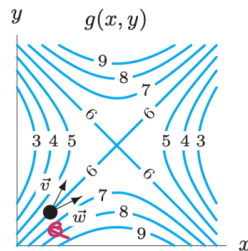
Summary

- ①  $\nabla f(a,b)$  points orthogonal to the level curve at  $(a,b)$
- ②  $\|\nabla f(a,b)\|$  is larger when the level curves are more tightly spaced near  $(a,b)$
- ③  $D_{\vec{u}} f(a,b) = 0$  when  $\vec{u}$  points tangent to the level curve at  $(a,b)$ .



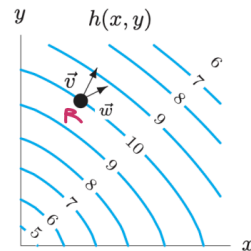
$$D_{\vec{u}} f(P) = 0$$

$$D_{\vec{w}} f(P) > 0$$



$$D_{\vec{u}} f(Q) < 0$$

$$D_{\vec{w}} f(Q) > 0$$

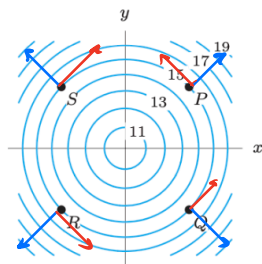


$$D_{\vec{u}} f(R) < 0$$

$$D_{\vec{w}} f(R) < 0.$$

Note:  
 $D_{\vec{u}} f(R) < D_{\vec{w}} f(R)$   
 $\uparrow$   
 more negative

**Problem 1.** In the contour plot below sketch the direction of  $\nabla f$  at each of the points  $P$ ,  $Q$ ,  $R$ , and  $S$ . Also sketch a direction  $\mathbf{u}$  where  $D_{\mathbf{u}}f$  is zero at each of these points.

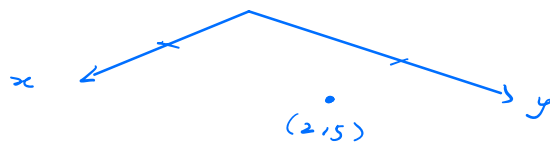
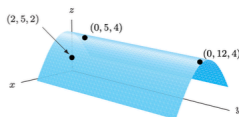


red vectors are tangent to level curve  $\Rightarrow D_{\mathbf{u}}f = 0$

blue vectors are perpendicular and facing direction of increase.

**Problem 2.** Consider the graph of a function  $f(x, y)$  shown below. Give sign of the following directional derivatives

- a.  $D_{\mathbf{u}}f(2, 5)$  where  $\mathbf{u} = \langle -1, 0 \rangle$  +
- b.  $D_{\mathbf{u}}f(2, 5)$  where  $\mathbf{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$  -
- c.  $D_{\mathbf{u}}f(0, 5)$  where  $\mathbf{u} = \langle 0, 1 \rangle$  0
- d.  $D_{\mathbf{u}}f(0, 12)$  where  $\mathbf{u} = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$  0



**Problem 3.** Let  $f(x, y) = -x^2y + xy^2 + xy$  and  $P = (2, 1)$ . Compute  $D_{\mathbf{u}}f(P)$  for each unit vector  $\mathbf{u}$  given below.

- $\mathbf{u}$  in the direction of  $\mathbf{v} = \langle 3, 4 \rangle$
- $\mathbf{u}$  in the direction from  $P$  to  $Q = (1, -1)$
- $\mathbf{u}$  in the direction of maximum rate of change
- $\mathbf{u}$  in the direction of minimum (ie. most negative) rate of change
- $\mathbf{u}$  in the direction perpendicular to  $\nabla f(P)$

$$\nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle$$

$$\nabla f(P) = \langle -2, 2 \rangle$$

$$\textcircled{a} \quad \vec{u} = \langle 3/5, 4/5 \rangle \quad D_{\vec{u}}f(P) = \nabla f(P) \cdot \vec{u} = -\frac{6}{5} + \frac{8}{5} = \frac{2}{5}$$

$$\textcircled{b} \quad \vec{PQ} = \langle -1, -2 \rangle, \quad \vec{u} = \frac{1}{\sqrt{5}} \langle -1, -2 \rangle, \quad D_{\vec{u}}f(P) = \frac{2}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

$$\textcircled{c} \quad D_{\vec{u}}f(P) = \|\nabla f(P)\| = \sqrt{8}$$

$$\textcircled{d} \quad D_{\vec{u}}f(P) = -\|\nabla f(P)\| = -\sqrt{8}$$

$$\textcircled{e} \quad D_{\vec{u}}f(P) = 0$$