12.7 Tangent Planes

Many planes arise as the graphs of linear functions of the form

$$z = L(x,y) = mx + ny + b$$

slope-intercept form of plane eq.

(compare with $y = mx + b$)

Notice $L_x(x,y) = m$ and $L_y(x,y) = n$ and $L(0,0) = b$,

so m is the slope of the plane in the x-direction

n is the slope of the plane in the y-direction

b is the z-axis intercept of the plane.

The point-slope form of a plane is written as

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

This plane passes through the point (x_0, y_0, z_0) with slope m in x-direction and slope n in y-direction

(compare with $y = y_0 + m(x - x_0)$ form of a line)

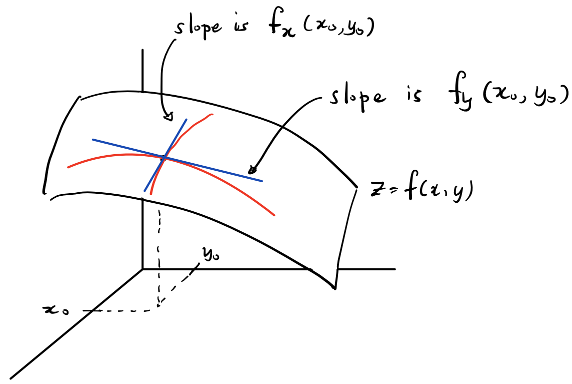
Examples Write equations for the following planes

① crosses z-axis at $z = 5$ and has x-direction slope 2, y-direction slope -1.

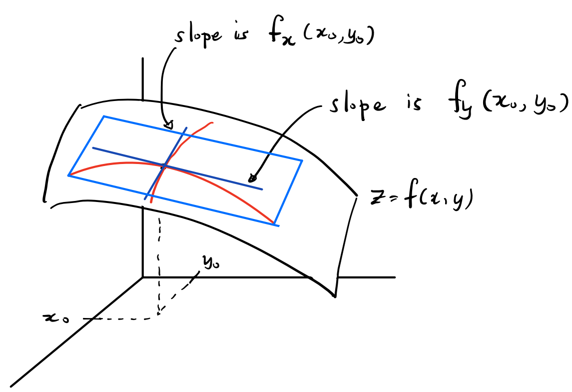
$z = 2x - y + 5$

② passes through the point $(1,2,3)$ and has x-direction slope -4, y-direction slope 6

$z = 3 - 4(x-1) + 6(x-2)$



The slope of the graph of $f(x,y)$ at (x_0,y_0) is $f_x(x_0,y_0)$ in the x-direction and is $f_y(x_0,y_0)$ in the y-direction.



Conclusion The formula for the tangent plane of the graph of f at $(x_0, y_0, f(x_0,y_0))$ is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

(point-slope form of tangent plane)

Example Find equation of tangent plane of $f(x,y) = 2x^2y$ at the point $(1,3)$ in point-slope form.

$$f(1,3) = 6$$

$$f_x = 4xy, \quad f_x(1,3) = 12 \qquad z = 6 + 12(x-1) + 2(y-3)$$

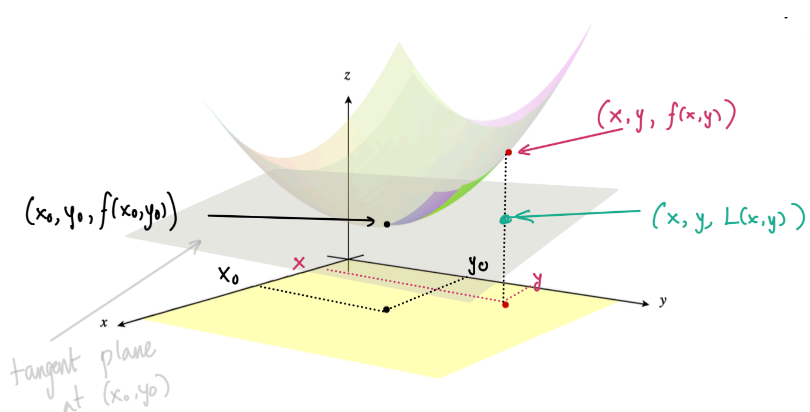
$$f_y = 2x^2 \quad f_y(1,3) = 2$$

Def The local linear approximation of a function $f(x,y)$ near (x_0,y_0) is the linear function $L(x,y)$ given by

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

tangent plane at (x_0,y_0)

When (x,y) is near (x_0,y_0), $f(x,y) \approx L(x,y)$.



Example Use the local linear approximation of $f(x,y) = 2x^2y$ near $(1,3)$ to approximate the value of $f(1.1, 3.2)$.

$L(x,y) = 6 + 12(x-1) + 2(y-3)$ (tangent plane found previously)

$$f(1.1, 3.2) \approx L(1.1, 3.2)$$
$$= 6 + 12(0.1) + 2(0.2)$$
$$= 6 + 1.2 + 0.4 = 7.6$$

(actual value is $2(1.1)^2(3.2) = 7.744$.)

Example The heat index I is a way to measure perceived temperature. It is a complicated function of the actual temperature T and the relative humidity H. So $I = f(T, H)$ and some function values are shown.

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

ⓐ Give the local linear approximation of f near $T = 96°$ and $H = 70\%$.

$f(96, 70) = 125$

$f_T(96,70) \approx \dfrac{\Delta I}{\Delta T} = \dfrac{133-125}{98-96} = 4$

$f_H(96,70) \approx \dfrac{\Delta I}{\Delta H} = \dfrac{125-121}{70-65} = 0.8$

$L(T, H) = 125 + 4(T-96) + 0.8(H-70)$

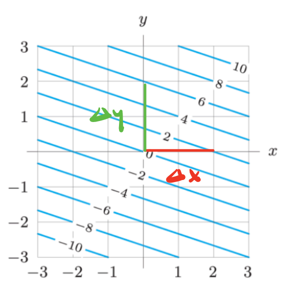
ⓑ Use your local linear approximation to estimate the heat index when $T = 97°$ and $H = 68\%$.

$f(97, 68) \approx L(97,68) = 125 + 4(97-96) + 0.8(68-70)$

$= 125 + 4 - 1.6$

$= 127.4$

Problem 1. The figure below shows the contour plot for a plane. Use it to find the slopes in the x and y directions and then write the equation of the plane in slope-intercept form.



$L_x \approx \dfrac{\Delta z}{\Delta x} = \dfrac{2-0}{2-0} = 1$

$L_y \approx \dfrac{\Delta z}{\Delta y} = \dfrac{6-0}{2-0} = 3$

$L(0,0) = 0.$

$z = x + 3y$

Problem 1. At a distance x feet from the beach, the price in thousands of dollars of a plot of land with area a square feet is given by $f(a, x)$. Suppose $f(1000, 300) = 200$, $f_a(1000, 300) = 3$, and $f_x(1000, 300) = -2$.

a. What are the units and practical meaning of $f(1000, 300)$?

b. What are the units and practical meaning of $f_a(1000, 300)$?

c. What are the units and practical meaning of $f_x(1000, 300)$?

d. Which is cheaper: 1005 square feet that are 305 feet from the beach or 998 square feet that are 295 feet from the beach? Justify your answer with tangent plane approximations.

(a) thousands of dollars, the cost of 1000 sq. ft. that is 300 ft. from the beach costs \$200,000

(b) thousands of dollars per square foot, the cost of land increases by \$3000 per square foot beyond 1000 sq. ft when the distance is held constant at 300 ft.

(c) thousands of dollars per square foot, the cost of land decreases by \$2000 per foot beyond 300 ft. from beach when the area is held constant at 1000 sq. ft.

(d) $f(x,y) \approx f(1000, 300) + f_a(1000, 300)\,(a - 1000) + f_x(1000, 300)\,(x - 300)$

$= 200 + 3(a - 1000) - 2(x - 300)$

so $f(1005, 305) \approx 200 + 3(5) - 2(5) = 205$

and $f(998, 295) \approx 200 + 3(-2) - 2(-5) = 204$

The second option seems to be cheaper.

Problem 2. The volume V of a certain gas is related to its temperature T and pressure P by the function $V = f(T, P)$. Suppose volume (in cubic inches) at temperature 500 degrees and pressure 24 (in pounds per square inch) is 23.69 cubic inches. If the pressure increases to 26 pounds per square inch and the temperature is held fixed at 500, the volume becomes 21.86 cubic inches. If the temperature increases to 520 and the pressure is held fixed at 24 pounds per square inch, the volume becomes 24.20 cubic inches.

a. Estimate $f_T(500, 24)$ and $f_P(500, 24)$ using the given information.

b. Estimate the value of $f(505, 24.3)$ using a tangent plane approximation of f.

(a) $$f_P(500, 24) \approx \frac{\Delta V}{\Delta P} = \frac{21.86 - 23.69}{26 - 24} = -0.915$$

$$f_T(500, 24) \approx \frac{\Delta V}{\Delta T} = \frac{24.20 - 23.69}{520 - 500} = 0.0255$$

(b) $$f(T, P) \approx f(500, 24) + f_T(500, 24)(T - 500) + f_P(500, 24)(P - 24)$$

$$= 23.69 + 0.0255(T - 500) - 0.915(P - 24)$$

so $$f(505, 24.3) \approx 23.69 + 0.0255(5) - 0.915(0.3) = 23.543$$

Problem 3. For each function f and point (x_0, y_0) below, find the equation of the tangent plane at $(x_0, y_0, f(x_0, y_0))$.

a. $f(x, y) = ye^{xy}, (x_0, y_0) = (1, 1)$

b. $f(x, y) = \sin(xy), (x_0, y_0) = (2, 3\pi/4)$

c. $f(x, y) = \ln(x^2 + 1) + y^2, (x_0, y_0) = (0, 3)$

(a) $f(1,1) = e$

$f_x = y^2 e^{xy}$, $f_x(1,1) = e$

$f_y = e^{xy} + xye^{xy}$, $f_y(1,1) = 2e$

$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= e + e(x-1) + 2e(y-1)$

(b) $f(2, \frac{3\pi}{4}) = \sin\left(\frac{3\pi}{2}\right) = -1$

$f_x = y\cos(xy)$, $f_x(2, \frac{3\pi}{4}) = \frac{3\pi}{4}\cos\left(\frac{3\pi}{2}\right) = 0$

$f_y = x\cos(xy)$, $f_y(2, \frac{3\pi}{4}) = 2\cos\left(\frac{3\pi}{2}\right) = 0$

$z = -1$

(c) $f(0,3) = \ln(1) + 9 = 9$

$f_x = \frac{2x}{x^2+1}$, $f_x(0,3) = 0$

$f_y = 2y$, $f_y(0,3) = 6$

$z = 9 + 6(y-3)$