12.7 Tangent Planes

A plane passing through (x_0, y_0, z_0) is given by an equation of the form

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

When $c \neq 0$, we can solve for z and get

$$z = z_0 - \frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0)$$

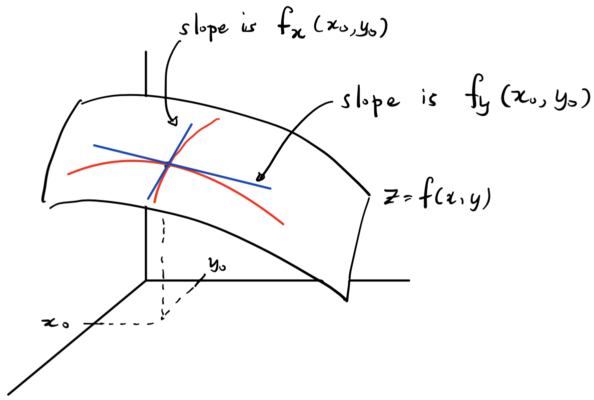
$$= z_0 + A(x-x_0) + B(y-y_0) \quad \text{(point-slope form of plane)}$$

(where $A = -\frac{a}{c}$, $B = \frac{-b}{c}$). Notice taking partial derivatives of the right side gives

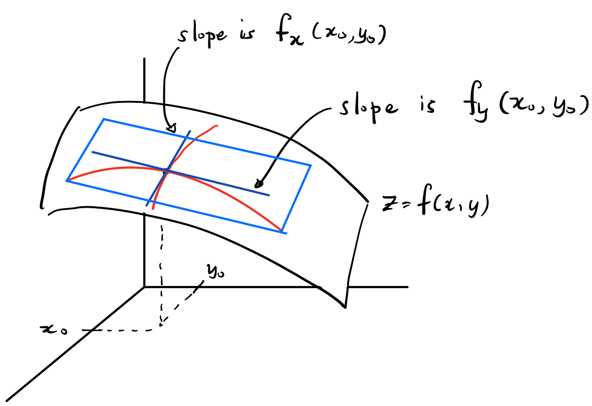
$$z_x = A \quad \text{and} \quad z_y = B.$$

So A represents the slope of the plane in the x-direction and B represents its slope in the y-direction.

Example $z = 3 - 4(x-1) + 6(y-2)$ passes through $(1,2,3)$ and has slope -4 in x-direction and 6 in y-direction.



The slope of the graph of $f(x,y)$ at (x_0,y_0) is $f_x(x_0,y_0)$ in the x-direction and is $f_y(x_0,y_0)$ in the y-direction.



Conclusion The formula for the tangent plane of the graph of f at $(x_0, y_0, f(x_0,y_0))$ is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)\left(x-x_0\right) + f_y(x_0,y_0)\left(y-y_0\right)$$

(point-slope form of tangent plane)

Example Find equation of tangent plane of $f(x,y) = 2x^2y$ at the point $(1,3)$ in point-slope form.

$f(1,3) = 6$

$f_x = 4xy$, $f_x(1,3) = 12$

$f_y = 2x^2$ $f_y(1,3) = 2$

$z = 6 + 12(x-1) + 2(y-3)$

Example Let $f(x,y) = -2x^2 - y^2 + xy + 4$ and use the tangent plane at $(1,1)$ to approximate $f(1.1, 1.1)$.

$f(1,1) = 2$

$f_x = -4x + y$, $f_x(1,1) = -3$

$f_y = -2y + x$, $f_y(1,1) = -1$

$z = 2 - 3(x-1) - (y-1)$

$z(1.1, 1.1) = 2 - 3(0.1) - (0.1)$

$= 1.6$

(Actual value of $f(1.1, 1.1)$ is 1.58)

Problem 1. At a distance x feet from the beach, the price in thousands of dollars of a plot of land with area a square feet is given by $f(a, x)$. Suppose $f(1000, 300) = 200$, $f_a(1000, 300) = 3$, and $f_x(1000, 300) = -2$.

a. What are the units and practical meaning of $f(1000, 300)$?

b. What are the units and practical meaning of $f_a(1000, 300)$?

c. What are the units and practical meaning of $f_x(1000, 300)$?

d. Which is cheaper: 1005 square feet that are 305 feet from the beach or 998 square feet that are 295 feet from the beach? Justify your answer with tangent plane approximations.

(a) thousands of dollars, the cost of 1000 sq. ft. that is 300 ft. from the beach costs \$200,000

(b) thousands of dollars per square foot, the cost of land increases by \$3000 per square foot beyond 1000 sq. ft when the distance is held constant at 300 ft.

(c) thousands of dollars per square foot, the cost of land decreases by \$2000 per foot beyond 300 ft. from beach when the area is held constant at 1000 sq. ft.

(d) $f(x,y) \approx f(1000, 300) + f_a(1000, 300)(a - 1000) + f_x(1000, 300)(x - 300)$

$$= 200 + 3(a - 1000) - 2(x - 300)$$

so $f(1005, 305) \approx 200 + 3(5) - 2(5) = 205$

and $f(998, 295) \approx 200 + 3(-2) - 2(-5) = 204$

The second option seems to be cheaper.

Problem 2. The volume V of a certain gas is related to its temperature T and pressure P by the function $V = f(T, P)$. Suppose volume (in cubic inches) at temperature 500 degrees and pressure 24 (in pounds per square inch) is 23.69 cubic inches. If the pressure increases to 26 pounds per square inch and the temperature is held fixed at 500, the volume becomes 21.86 cubic inches. If the temperature increases to 520 and the pressure is held fixed at 24 pounds per square inch, the volume becomes 24.20 cubic inches.

a. Estimate $f_T(500, 24)$ and $f_P(500, 24)$ using the given information.

b. Estimate the value of $f(505, 24.3)$ using a tangent plane approximation of f.

(a) $$f_P(500,24) \approx \frac{\Delta V}{\Delta P} = \frac{21.86 - 23.69}{26 - 24} = -0.915$$

$$f_T(500,24) \approx \frac{\Delta V}{\Delta T} = \frac{24.20 - 23.69}{520 - 500} = 0.0255$$

(b) $$f(T,P) \approx f(500,24) + f_T(500,24)(T-500) + f_P(500,24)(P-24)$$

$$= 23.69 + 0.0255(T-500) - 0.915(P-24)$$

so $$f(505, 24.3) \approx 23.69 + 0.0255(5) - 0.915(0.3) = 23.543$$

Problem 3. For each function f and point (x_0, y_0) below, find the equation of the tangent plane at $(x_0, y_0, f(x_0, y_0))$.

a. $f(x, y) = ye^{xy}, (x_0, y_0) = (1, 1)$

b. $f(x, y) = \sin(xy), (x_0, y_0) = (2, 3\pi/4)$

c. $f(x, y) = \ln(x^2 + 1) + y^2, (x_0, y_0) = (0, 3)$

(a) $f(1,1) = e$

$f_x = y^2 e^{xy}, \quad f_x(1,1) = e$

$f_y = e^{xy} + xye^{xy}, \quad f_y(1,1) = 2e$

$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= e + e(x-1) + 2e(y-1)$

(b) $f(2, \frac{3\pi}{4}) = \sin\left(\frac{3\pi}{2}\right) = -1$

$f_x = y\cos(xy), \quad f_x(2, \frac{3\pi}{4}) = \frac{3\pi}{4}\cos\left(\frac{3\pi}{2}\right) = 0$

$f_y = x\cos(xy), \quad f_y(2, \frac{3\pi}{4}) = 2\cos\left(\frac{3\pi}{2}\right) = 0$

$z = -1$

(c) $f(0,3) = \ln(1) + 9 = 9$

$f_x = \frac{2x}{x^2+1}, \quad f_x(0,3) = 0$

$f_y = 2y, \quad f_y(0,3) = 6$

$z = 9 + 6(y-3)$