Problem 1. Consider the vectors $\mathbf{v} = \langle 3, 2, -2\rangle$ and $\mathbf{w} = \langle 4, -3, 1\rangle$.

a. Compute $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$.

b. Find the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

c. Which option is true: (i) $\theta = 0$, (ii) $0 < \theta < \pi/2$, (iii) $\theta = \pi/2$, (iv) $\pi/2 < \theta < \pi$, (v) $\theta = \pi$?

d. Find the unit vector in the direction opposite of $\mathbf{v}$.

e. Find a vector of length 5 in the direction of $\mathbf{w}$.

f. Make a general sketch, not specific to these vectors, of a parallelogram using $\mathbf{v}$ and $\mathbf{w}$ and label the two diagonals of the parallelogram with $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ appropriately.

g. Find the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$.

h. Find the area of the triangle formed by the vectors $\mathbf{v}, \mathbf{w}, \mathbf{v} - \mathbf{w}$.

i. Make a general sketch, not specific to these vectors, of $\mathbf{v}$, $\mathbf{w}$, and the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$.

j. Find the orthogonal projection $\mathbf{p}$ of $\mathbf{v}$ onto $\mathbf{w}$.

k. Suppose $\mathbf{q} = \mathbf{v} - \mathbf{p}$. Find $\mathbf{q} \cdot \mathbf{w}$.

l. Find a standard form equation of the plane containing $P = (1, 2, 3)$ and parallel to both $\mathbf{v}$ and $\mathbf{w}$.

m. Find a vector equation of the line containing P and orthogonal to the plane in the previous part.

(a) $\vec{v} \cdot \vec{w} = 12 - 6 - 2 = 4$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -2 \\ 4 & -3 & 1 \end{vmatrix} = \langle -4, -11, -17\rangle$$

(b) $\|\vec{v}\| = \sqrt{17}$, $\|\vec{w}\| = \sqrt{26}$,

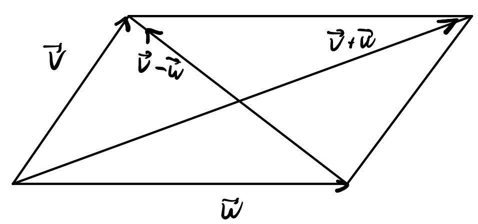
$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{4}{\sqrt{17}\sqrt{26}}$$

(c) since $\cos\theta > 0$, only (i) and (ii) are possible,

and since $\vec{v}$ and $\vec{w}$ are not parallel, it must be (ii)

(d) $-\frac{1}{\sqrt{17}}\langle 3, 2, -2\rangle$ (e) $\frac{5}{\sqrt{26}}\langle 4, -3, 1\rangle$

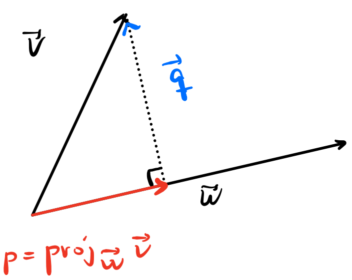
(f)



(g) $\|\vec{v}\times\vec{w}\| = \sqrt{16 + 121 + 289} = \sqrt{426}$

(h) $\frac{1}{2}\sqrt{426}$

(i)



(j) $$\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{4}{26}\langle 4, -3, 1\rangle$$

$$= \left\langle \frac{8}{13}, \frac{-6}{13}, \frac{2}{13}\right\rangle$$

(k) $\vec{q}\cdot\vec{w} = 0$

(l) $\vec{n} = \vec{v}\times\vec{w} = \langle -4, -11, -17\rangle$

$-4(x-1) - 11(y-2) - 17(z-3) = 0.$

(m) $\ell(t) = \langle 1, 2, 3\rangle + t\langle -4, -11, -17\rangle$

Problem 2. Consider the line ℓ_1 passing through the point $(1,2,4)$ in the direction of $\langle 3,1,-1\rangle$, the line $\ell_2(t) = \langle 1,1,3\rangle + t\,\langle 3,1,2\rangle$, the plane Π_1 with equation $5(x-1)+2y-3(z+1)=0$ and the plane Π_2 with equation $3x+y+2z=10$.

a. Determine whether ℓ_1 and ℓ_2 are parallel, intersect, or are skew lines.

b. Determine whether ℓ_1 is orthogonal to Π_1. Do the same with Π_2.

c. Repeat the previous question with ℓ_2.

d. Determine whether ℓ_1 is parallel to Π_1. (In other words, does ℓ_1 lie on Π_1 or could it be translated so that it lies on Π_1?) Do the same with Π_2.

e. Repeat the previous question with ℓ_2.

f. If ℓ_1 intersects Π_1 find the point of intersection. Do the same with Π_2.

g. Determine whether Π_1 and Π_2 are parallel.

a) $\langle 3,1,-1\rangle$ and $\langle 3,1,2\rangle$ are not parallel so ℓ_1 and ℓ_2 are not parallel

Do they intersect? If so, there exist, s and t so that

$$\begin{cases} 1+3t = 1+3s & \Rightarrow s=t \\ 2+t = 1+s & \Rightarrow 2+t = 1+t \\ 4-t = 3+2s & \Rightarrow 2=1 \text{, a contradiction} \end{cases}$$

They are skew lines.

(b) ℓ_1 is orthogonal to Π_1 if $\langle 3,1,-1\rangle$ is parallel to normal vector for Π_1, $\langle 5,2,-3\rangle$ (No)

ℓ_1 is not orthogonal to Π_2 since $\langle 3,1,-1\rangle$ is not parallel to normal vector for Π_2, $\langle 3,1,2\rangle$

(c) ℓ_2 <u>is not</u> orthogonal to Π_1, <u>is</u> orthogonal to Π_2

(d) ℓ_1 is parallel to Π_1 if $\langle 3,1,-1\rangle$ is orthogonal to $\langle 5,2,-3\rangle$: $\langle 3,1,-1\rangle \cdot \langle 5,2,-3\rangle \neq 0$

not parallel to Π_1

$\langle 3,1,-1\rangle \cdot \langle 3,1,2\rangle \neq 0 \Rightarrow$ not parallel to Π_2

(e) $\langle 3,1,2\rangle \cdot \langle 5,2,-3\rangle \neq 0 \Rightarrow l_2$ not parallel to Π_1

$\langle 3,1,2\rangle \cdot \langle 3,1,2\rangle \neq 0 \Rightarrow l_2$ not parallel to Π_2

(f) intersection of l_1 with Π_1 :

find t so that $\langle 1,2,4\rangle + t\langle 3,1,-1\rangle$ satisfies

$5(x-1) + 2y - 3(z+1) = 0$:

$$5\big((1+3t)-1\big) + 2(2+t) - 3\big((4-t)+1\big) = 0$$

$$\Rightarrow 15t + 4 + 2t - 15 + 3t = 0 \Rightarrow 20t = 11$$

$$\Rightarrow t = 11/20$$

point of intersection : $\left(1+\frac{33}{20},\ 2+\frac{11}{20},\ 4-\frac{11}{20}\right)$

$$= \left(\frac{53}{20},\ \frac{51}{20},\ \frac{69}{20}\right)$$

find t so that $\langle 1,1,3\rangle + t\langle 3,1,2\rangle$ satisfies

$3x + y + 2z = 10$:

$$3(1+3t) + (1+t) + 2(3+2t) = 10$$

$$\Rightarrow 3 + 9t + 1 + t + 6 + 4t = 10$$

$$\Rightarrow 14t = 0 \Rightarrow t = 0$$

point of intersection : $(1,1,3)$

(g) not parallel since normal vectors not parallel.

Problem 3. Sketch the surfaces determine by the following equations and give the technical term for the shape (eg. *circular cylinder*).

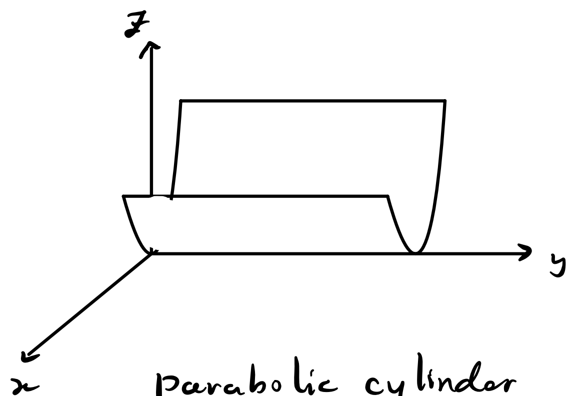
a. $z = x^2$

b. $y^2 + z^2 = 1$

c. $z = 1$

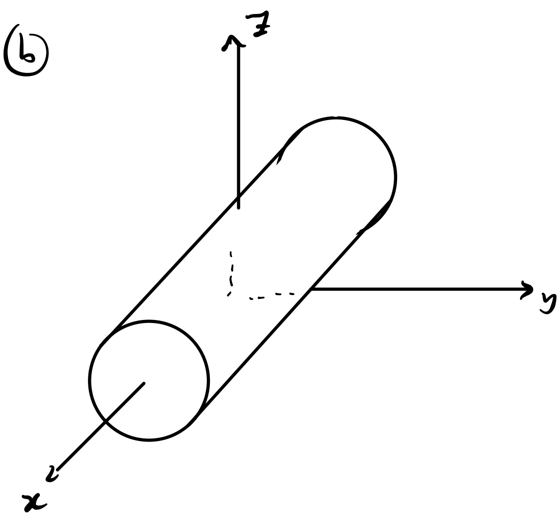
d. $x^2 + y^2 + z^2 = 4$

(a)



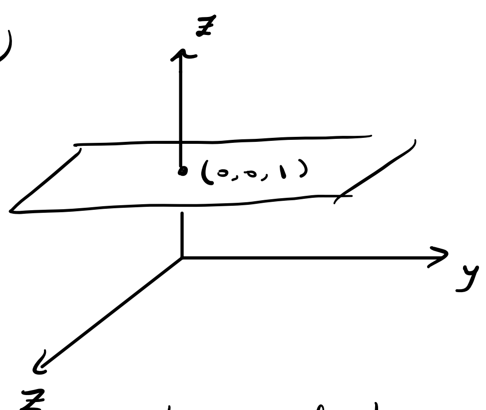
parabolic cylinder

(b)



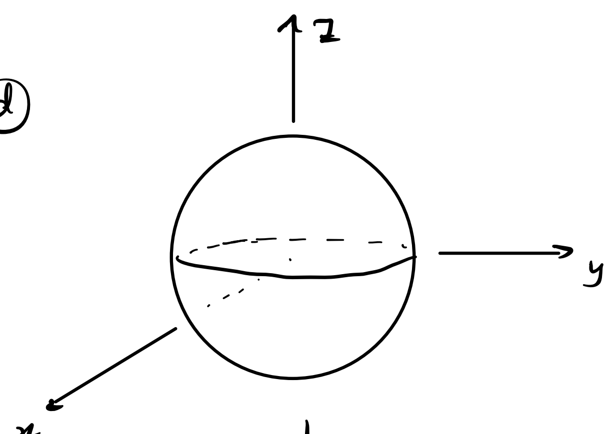
circular cylinder

(c)



horizontal plane
at height 1.

(d)



sphere

Problem 4. Let $\mathbf{r}(t) = \langle 1 + 3\cos(-t), 2 + 3\sin(-t)\rangle$ for $0 \le t \le 2\pi$.

a. Describe in words and plot the curve traced by $\mathbf{r}(t)$.

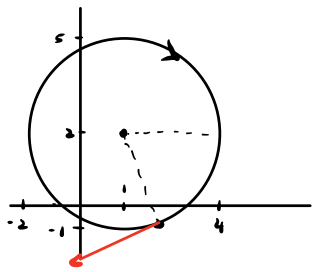
b. Compute $\mathbf{r}'(t)$.

c. Find the tangent vector of $\mathbf{r}(t)$ when $t = \pi/3$ and plot it in your sketch above.

d. Find a vector equation of the tangent line to the curve when $t = \pi/3$.

e. Set up and compute an integral for the length of the curve traced by $\mathbf{r}(t)$.

(a) circle with radius 3, centered at (1, 2), traced clockwise starting from (4, 2)



(b) $\vec{r}\,'(t) = \langle 3\sin(-t), -3\cos(-t)\rangle$

(c) $\vec{r}\,'(\frac{\pi}{3}) = \langle 3\sin(\frac{-\pi}{3}), -3\cos(\frac{-\pi}{3})\rangle$

$= \langle \frac{-3\sqrt{3}}{2}, \frac{-3}{2}\rangle$

(d) $\vec{r}(\frac{\pi}{3}) = \langle 1 + 3\cos(\frac{-\pi}{3}), 2 + 3\sin(\frac{-\pi}{3})\rangle$

$= \langle 1 + \frac{3}{2}, 2 - \frac{3\sqrt{3}}{2}\rangle = \langle \frac{5}{2}, \frac{4-3\sqrt{3}}{2}\rangle$

$l(t) = \langle \frac{5}{2}, \frac{4-3\sqrt{3}}{2}\rangle + t\langle \frac{-3\sqrt{3}}{2}, \frac{-3}{2}\rangle$

(e) $\int_0^{2\pi} \|\vec{r}\,'(t)\|\,dt = \int_0^{2\pi} \sqrt{(3\sin(-t))^2 + (3\cos(-t))^2}\,dt = \int_0^{2\pi} 3\,dt$

$= 6\pi$

Problem 5. Compute $\mathbf{r}'(t)$ for each example below.

a. $\mathbf{r}(t) = \left\langle e^{t^2}, \sin(t^3 + 3t^2), \cos(e^{5t}) \right\rangle$

b. $\mathbf{r}(t) = \langle t \ln t, t^2 \sin(4t), t^3 e^{2t} \rangle$

c. $\mathbf{r}(t) = \left\langle \frac{e^{2t}}{t^3+t^2}, \frac{t^4}{\sin(2t)+\cos t}, \frac{t^5+t^3}{t^2+1} \right\rangle$

(a) $\vec{r}\,'(t) = \left\langle 2te^{t^2}, (3t^2 + 6t)\cos(t^3 + 3t^2), -5e^{5t}\sin(e^{5t}) \right\rangle$

(b) $\vec{r}\,'(t) = \left\langle \ln t + 1, 2t\sin(4t) + 4t^2\cos(4t), 3t^2e^{2t} + 2t^3e^{2t} \right\rangle$

(c) $$\vec{r}\,'(t) = \left\langle \frac{2e^{2t}(t^3+t^2) - e^{2t}(3t^2+2t)}{(t^3+t^2)^2}, \frac{4t^3(\sin(2t)+\cos t) - t^4(2\cos(2t) - \sin t)}{(\sin(2t)+\cos t)^2}, \frac{(5t^4+3t^2)(t^2+1) - 2t(t^5+t^3)}{(t^2+1)^2} \right\rangle$$