Problem 1. Consider the vectors $\mathbf{v} = \langle 3, 2, -2\rangle$ and $\mathbf{w} = \langle 4, -3, 1\rangle$.

a. Compute $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$.

b. Find the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

c. Which option is true: (i) $\theta = 0$, (ii) $0 < \theta < \pi/2$, (iii) $\theta = \pi/2$, (iv) $\pi/2 < \theta < \pi$, (v) $\theta = \pi$?

d. Find the unit vector in the direction opposite of $\mathbf{v}$.

e. Find a vector of length 5 in the direction of $\mathbf{w}$.

f. Make a general sketch, not specific to these vectors, of a parallelogram using $\mathbf{v}$ and $\mathbf{w}$ and label the two diagonals of the parallelogram with $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ appropriately.

g. Find the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$.

h. Find the area of the triangle formed by the vectors $\mathbf{v}, \mathbf{w}, \mathbf{v}-\mathbf{w}$.

i. Make a general sketch, not specific to these vectors, of $\mathbf{v}$, $\mathbf{w}$, and the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$.

j. Find the orthogonal projection $\mathbf{p}$ of $\mathbf{v}$ onto $\mathbf{w}$.

k. Suppose $\mathbf{q} = \mathbf{v} - \mathbf{p}$. Find $\mathbf{q} \cdot \mathbf{w}$.

l. Find a standard form equation of the plane containing $P = (1, 2, 3)$ and parallel to both $\mathbf{v}$ and $\mathbf{w}$.

m. Find a vector equation of the line containing P and orthogonal to the plane in the previous part.

(a) $\vec{v} \cdot \vec{w} = 12 - 6 - 2 = 4$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -2 \\ 4 & -3 & 1 \end{vmatrix} = \langle -4, -11, -17\rangle$$

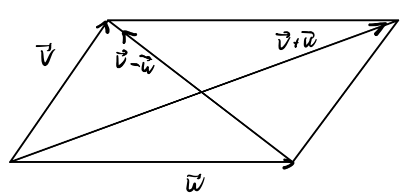
(b) $\|\vec{v}\| = \sqrt{17}$, $\|\vec{w}\| = \sqrt{26}$,

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{4}{\sqrt{17}\sqrt{26}}$$

(c) since $\cos\theta > 0$, only (i) and (ii) are possible, and since $\vec{v}$ and $\vec{w}$ are not parallel, it must be (ii)

(d) $-\frac{1}{\sqrt{17}}\langle 3, 2, -2\rangle$ (e) $\frac{5}{\sqrt{26}}\langle 4, -3, 1\rangle$

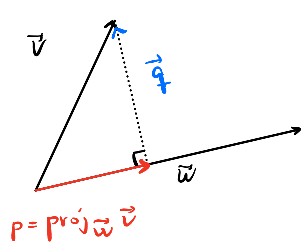
(f)



(g) $\|\vec{v} \times \vec{w}\| = \sqrt{16 + 121 + 289} = \sqrt{426}$

(h) $\frac{1}{2}\sqrt{426}$

(i)



(j) $\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{4}{26}\langle 4, -3, 1\rangle$

$= \left\langle \frac{8}{13}, \frac{-6}{13}, \frac{2}{13}\right\rangle$

(k) $\vec{q}\cdot\vec{w} = 0$

(l) $\vec{n} = \vec{v} \times \vec{w} = \langle -4, -11, -17\rangle$

$-4(x-1) - 11(y-2) - 17(z-3) = 0.$

(m) $\ell(t) = \langle 1, 2, 3\rangle + t\langle -4, -11, -17\rangle$

Problem 2. Consider the line ℓ_1 passing through the point $(1,2,4)$ in the direction of $\langle 3,1,-1\rangle$, the line $\ell_2(t) = \langle 1,1,3\rangle + t\,\langle 3,1,2\rangle$, the plane Π_1 with equation $5(x-1)+2y-3(z+1)=0$ and the plane Π_2 with equation $3x+y+2z=10$.

a. Determine whether ℓ_1 and ℓ_2 are parallel, intersect, or are skew lines.

b. Determine whether ℓ_1 is orthogonal to Π_1. Do the same with Π_2.

c. Repeat the previous question with ℓ_2.

d. Determine whether ℓ_1 is parallel to Π_1. Do the same with Π_2.

e. Repeat the previous question with ℓ_2.

f. If ℓ_1 intersects Π_1 find the point of intersection. Do the same with Π_2.

g. Repeat the previous question with ℓ_2.

h. Determine whether Π_1 and Π_2 are parallel.

a) $\langle 3,1,-1\rangle$ and $\langle 3,1,2\rangle$ are not parallel so ℓ_1 and ℓ_2 are not parallel

Do they intersect? If so, there exist, s and t so that

$$\begin{cases} 1+3t = 1+3s & \Rightarrow s=t \\ 2+t = 1+s & \Rightarrow 2+t = 1+t \\ 4-t = 3+2s & \Rightarrow 2=1, \text{ a contradiction} \end{cases}$$

They are skew lines.

(b) ℓ_1 is orthogonal to Π_1 if $\langle 3,1,-1\rangle$ is parallel to normal vector for Π_1, $\langle 5,2,3\rangle$ (No)

ℓ_1 is not orthogonal to Π_2 since $\langle 3,1,-1\rangle$ is not parallel to normal vector for Π_2, $\langle 3,1,2\rangle$

(c) ℓ_2 is not orthogonal to Π_1, is orthogonal to Π_2

(d) ℓ_1 is parallel to Π_1 if $\langle 3,1,-1\rangle$ is orthogonal to $\langle 5,2,3\rangle$: $\langle 3,1,-1\rangle \cdot \langle 5,2,3\rangle \neq 0$

not parallel to Π_1

$\langle 3,1,-1\rangle \cdot \langle 3,1,2\rangle \neq 0 \Rightarrow$ not parallel to Π_2

(e) $\langle 3,1,2\rangle \cdot \langle 5,2,3\rangle \neq 0 \Rightarrow \ell_2$ not parallel to Π_1

$\langle 3,1,2\rangle \cdot \langle 3,1,2\rangle \neq 0 \Rightarrow \ell_2$ not parallel to Π_2

(f) intersection of ℓ_1 with Π_1 :

find t so that $\langle 1,2,4\rangle + t\langle 3,1,-1\rangle$ satisfies

$5(x-1)+2y-3(z+1)=0$:

$$5\big((1+3t)-1\big)+2(2+t)-3\big((4-t)+1\big)=0$$

$$\Rightarrow 15t+4+2t-15+3t=0 \Rightarrow 20t=11$$

$$\Rightarrow t=11/20$$

point of intersection : $\left(1+\frac{33}{20},\ 2+\frac{11}{20},\ 4-\frac{11}{20}\right)$

$$=\left(\frac{53}{20},\ \frac{51}{20},\ \frac{69}{20}\right)$$

find t so that $\langle 1,1,3\rangle + t\langle 3,1,2\rangle$ satisfies

$3x+y+2z=10$:

$$3(1+3t)+(1+t)+2(3+2t)=10$$

$$\Rightarrow 3+9t+1+t+6+4t=10$$

$$\Rightarrow 14t=0 \Rightarrow t=0$$

point of intersection : $(1,1,3)$

(g) not parallel since normal vectors not parallel.

Problem 3. Sketch the surfaces determine by the following equations and give the technical term for the shape (eg. *circular cylinder*).

a. $z = 4 - (x^2 + y^2)$

b. $z = 1 + \sqrt{(x-1)^2 + y^2}$

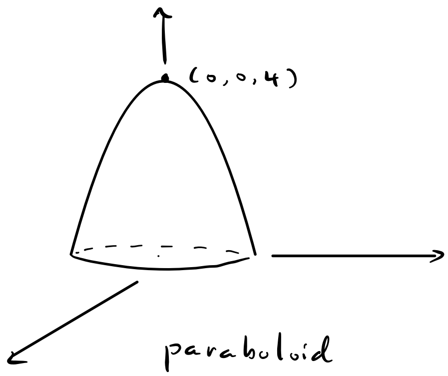
c. $z = x^2$

d. $y^2 + z^2 = 1$

e. $x + y + z = 1$

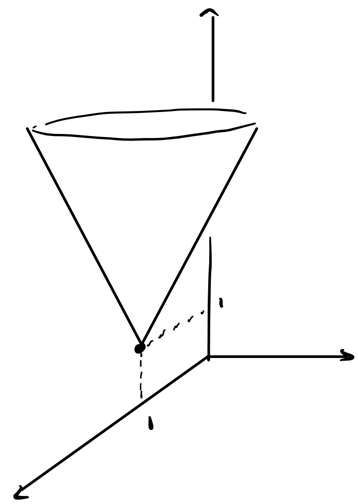
f. $x^2 + y^2 + z^2 = 4$

(a)



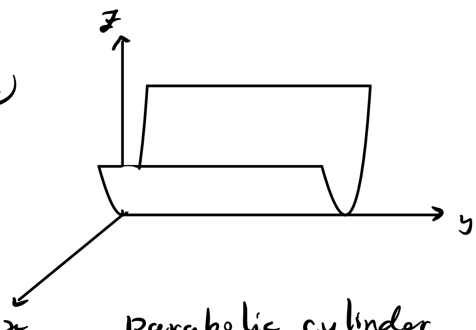
paraboloid

(b)



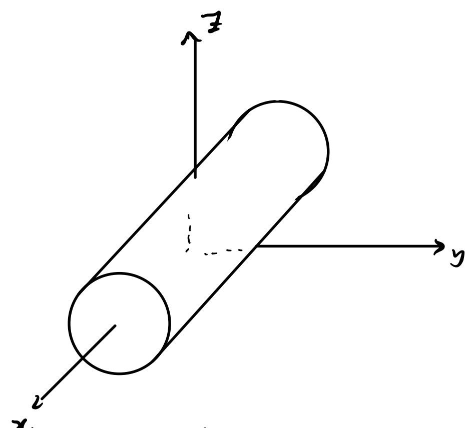
cone

(c)



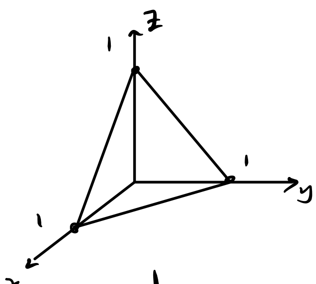
parabolic cylinder

(d)



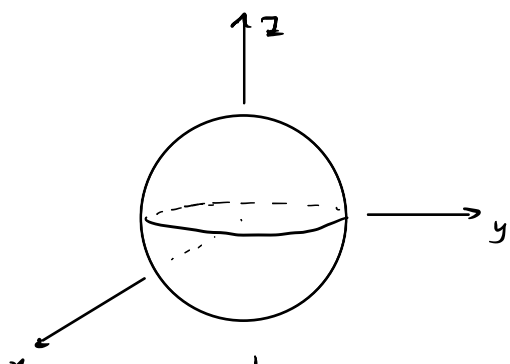
circular cylinder

(e)



plane

(f)



sphere

Problem 4. Consider the function $f(x, y) = 4 - (x-1)^2 - (y-2)^2$.

a. What are the domain and range of f?

b. Make a contour diagram using the $z = 0, 1, 2, 3, 4$ level curves.

c. Explain why the contour diagram cannot have any level curves of the form $f(x, y) = c$ where $c > 4$.

d. Sketch the graph of f in three dimensional space, taking note of how it relates to the contour diagram.

(a) domain is $\mathbb{R}^2$, range is $(-\infty, 4]$

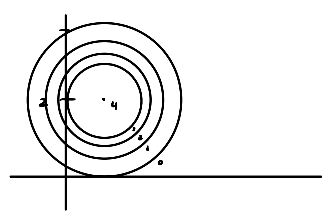
(b) level curves are circles of the form

$$c = 4 - (x-1)^2 - (y-2)^2$$

$$(x-1)^2 + (y-2)^2 = 4 - c$$

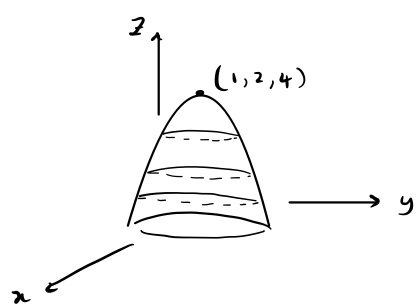
center $(1,2)$,
radius $\sqrt{4-c}$

c	radius
0	2
1	$\sqrt{3}$
2	$\sqrt{2}$
3	1
4	0



(c) Since the range of f is $(-\infty, 4]$, the output (or c-values) cannot exceed 4.

(d)



Problem 5. Let $\mathbf{r}(t) = \langle 1 + 3\cos(-t), 2 + 3\sin(-t)\rangle$ for $0 \le t \le 2\pi$.

a. Describe in words and plot the curve traced by $\mathbf{r}(t)$.

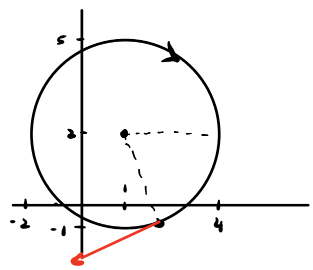
b. Compute $\mathbf{r}'(t)$.

c. Find the tangent vector of $\mathbf{r}(t)$ when $t = \pi/3$ and plot it in your sketch above.

d. Find a vector equation of the tangent line to the curve when $t = \pi/3$.

(a) circle with radius 3, centered at (1, 2),

traced clockwise starting from (4, 2)



(b) $\vec{r}\,'(t) = \langle 3\sin(-t), -3\cos(-t)\rangle$

(c) $\vec{r}\,'\left(\frac{\pi}{3}\right) = \left\langle 3\sin\left(\frac{-\pi}{3}\right), -3\cos\left(-\frac{\pi}{3}\right)\right\rangle$

$= \left\langle \frac{-3\sqrt{3}}{2}, \frac{-3}{2} \right\rangle$

(d) $\vec{r}\left(\frac{\pi}{3}\right) = \left\langle 1 + 3\cos\left(\frac{-\pi}{3}\right), 2 + 3\sin\left(\frac{-\pi}{3}\right)\right\rangle$

$= \left\langle 1 + \frac{3}{2}, 2 - \frac{3\sqrt{3}}{2} \right\rangle = \left\langle \frac{5}{2}, \frac{4-3\sqrt{3}}{2} \right\rangle$

$l(t) = \left\langle \frac{5}{2}, \frac{4-3\sqrt{3}}{2} \right\rangle + t\left\langle \frac{-3\sqrt{3}}{2}, \frac{-3}{2} \right\rangle$

Problem 6. Compute $\mathbf{r}'(t)$ for each example below.

a. $\mathbf{r}(t) = \left\langle e^{t^2}, \sin(t^3 + 3t^2), \cos(e^{5t}) \right\rangle$

b. $\mathbf{r}(t) = \langle t \ln t, t^2 \sin(4t), t^3 e^{2t} \rangle$

c. $\mathbf{r}(t) = \left\langle \frac{e^{2t}}{t^3+t^2}, \frac{t^4}{\sin(2t)+\cos t}, \frac{t^5+t^3}{t^2+1} \right\rangle$

ⓐ $\vec{r}\,'(t) = \left\langle 2te^{t^2}, (3t^2+6t)\cos(t^3+3t^2), -5e^{5t}\sin(e^{5t}) \right\rangle$

ⓑ $\vec{r}\,'(t) = \left\langle \ln t + 1, 2t\sin(4t) + 4t^2\cos(4t), 3t^2e^{2t} + 2t^3e^{2t} \right\rangle$

ⓒ $$\vec{r}\,'(t) = \left\langle \frac{2e^{2t}(t^3+t^2) - e^{2t}(3t^2+2t)}{(t^3+t^2)^2}, \right.$$

$$\frac{4t^3(\sin(2t)+\cos t) - t^4(2\cos(2t) - \sin t)}{(\sin(2t)+\cos t)^2},$$

$$\left. \frac{(5t^4+3t^2)(t^2+1) - 2t(t^5+t^3)}{(t^2+1)^2} \right\rangle$$