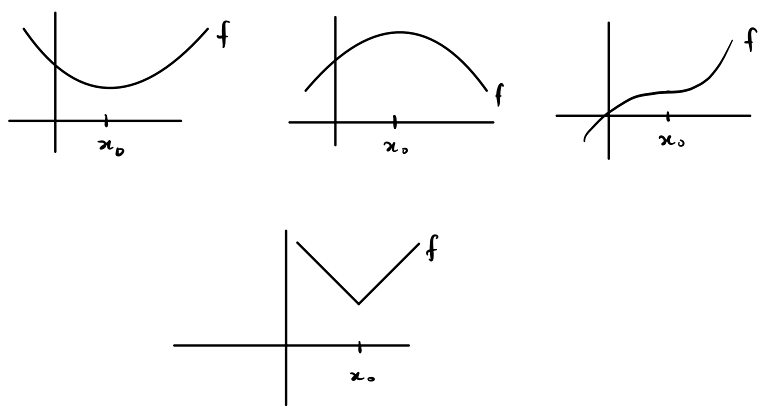
12.8 Extreme Values (ie. optimization)

Calc I reminders

A point x_0 is a critical point of f if

$f'(x_0) = 0$ or $f'(x_0)$ does not exist

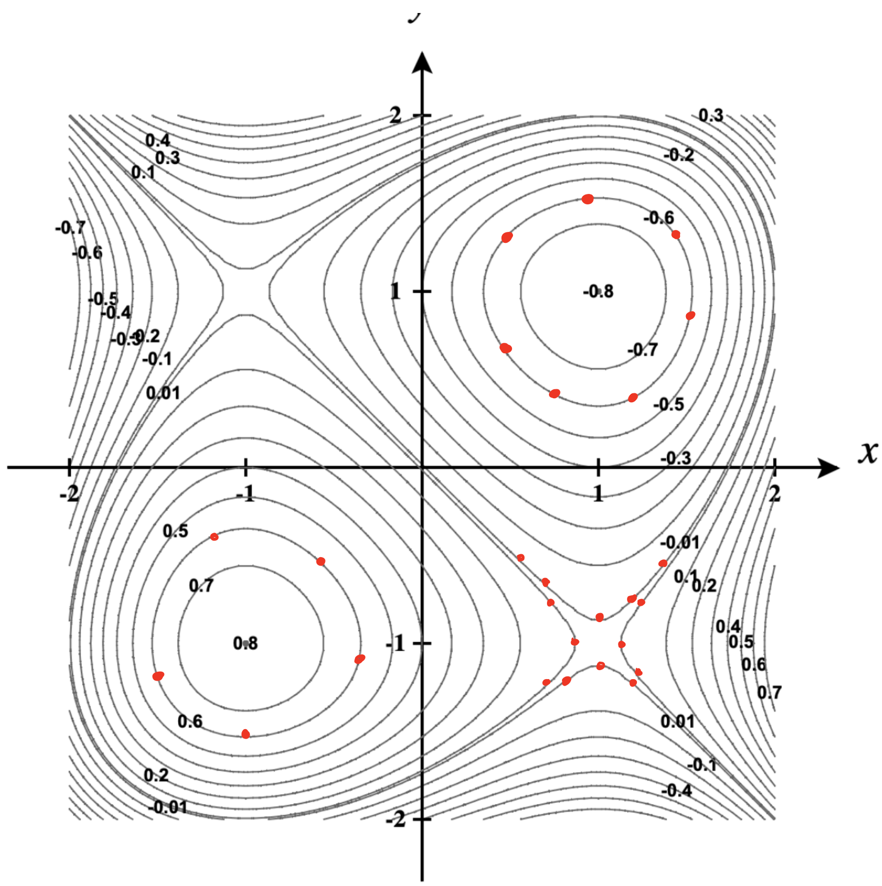
eg.



Second derivative test if x_0 is critical point and $f'(x_0) = 0$, then

1. if $f''(x_0) > 0$, local min occurs at x_0
2. if $f''(x_0) < 0$, local max occurs at x_0
3. if $f''(x_0) = 0$, test is inconclusive.

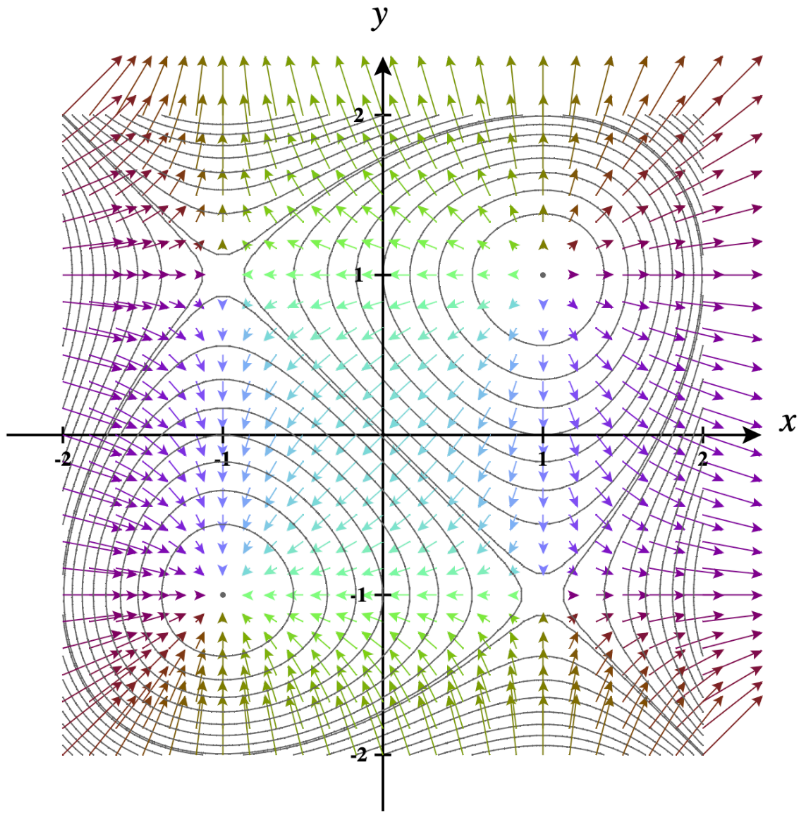
Warm up questions The contour plot below is for a function we'll call f



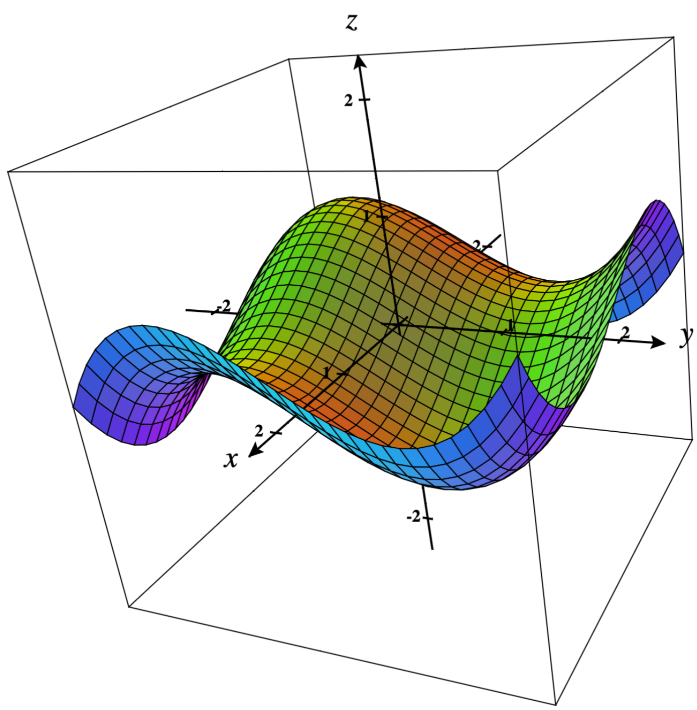
① At each red point above, which direction do the gradient vectors of f point?

② How do the lengths of the gradient vectors change as we move away from $(1,1)$ and $(-1,-1)$? Think about spacing of the level curves.

③ There are 4 "critical points". Which do you think are the locations of local min and local max?



gradient vector field is shown



the graph of f

Def A point (x_0, y_0) in the domain of $f(x,y)$

① is a critical point of f if $\nabla f(x_0, y_0) = \vec{0}$

or $\nabla f(x_0, y_0)$ is undefined.

② is a local min if $f(x_0, y_0) \le f(x,y)$ for all

(x, y) near (x_0, y_0)

③ is a local max if $f(x_0, y_0) \ge f(x,y)$ for all (x,y)

near (x_0, y_0).

④ is a saddle point if it's a critical point

and for any region around (x_0, y_0) there exists

(x_1, y_1) and (x_2, y_2) in that region so that

$$f(x_1, y_1) < f(x_0, y_0) < f(x_2, y_2)$$

Theorem (Second Derivative Test) Let (x_0, y_0) be

a critical point with $\nabla f(x_0, y_0) = \vec{0}$ and let

$$D = f_{xx} f_{yy} - f_{xy}^2 \quad \text{(called the discriminant).}$$

Then ① if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then

(x_0, y_0) is a local min.

② if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then

(x_0, y_0) is a local max.

③ if $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

④ if $D(x_0, y_0) = 0$, then the test is inconclusive.

Remark $D(x_0,y_0)>0$ is a condition that $D_{\vec{u}}(D_{\vec{u}} f(x_0,y_0))$ has the same sign (positive or negative) in every direction $\vec{u}$ (guarantees concavity agrees in every direction).

Proof of this requires linear algebra.

Example Let $f(x,y) = \frac{1}{5}(x^3+y^3-3x-3y)$. Find all critical points of f and classify them with the Second Derivative Test.

$$\nabla f = \frac{1}{5}\langle 3x^2-3,\ 3y^2-3\rangle$$

$\nabla f = \vec{0}$ means $\begin{cases} \frac{1}{5}(3x^2-3)=0 \\ \frac{1}{5}(3y^2-3)=0 \end{cases} \Rightarrow \begin{matrix} x=\pm 1 \\ y=\pm 1 \end{matrix}$

So $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$ are the critical points of f.

$f_{xx}=6x$, $f_{yy}=6y$, $f_{xy}=0$, so $D(x,y)=36xy$

point	D	f_{xx}	type
$(1,1)$	36	6	local min
$(1,-1)$	-36		saddle point
$(-1,1)$	-36		saddle point
$(-1,-1)$	36	-6	local max

Problem 1. Find all critical points of the following functions and use the Second Derivative Test to classify them.

a. $f(x,y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$

b. $f(x,y) = 4 + x^3 + y^3 - 3xy$

c. $f(x,y) = x^2 + y^3 - 3y + 1$

(a) $\nabla f = \langle x+4, 4y-8 \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} x+4=0 \\ 4y-8=0 \end{cases} \Rightarrow \begin{matrix} x=-4 \\ y=2 \end{matrix} \Rightarrow (-4,2) \text{ is the only critical point}$$

$f_{xx} = 1$, $f_{yy} = 4$, $f_{xy} = 0 \Rightarrow D(x,y) = 4$

$\Rightarrow D(-4,2) = 4 > 0$, $f_{xx}(-4,2) = 1 > 0$

$\Rightarrow (-4,2)$ is local min

(b) $\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} y = x^2 \\ x = y^2 \end{cases} \Rightarrow x = x^4 \Rightarrow x^4 - x = 0$$

$\Rightarrow x(x^3 - 1) = 0$

$\Rightarrow x = 0, \quad x = 1$

So $(0,0)$ and $(1,1)$ are the critical points

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3 \Rightarrow D(x,y) = 36xy - 9$

point	D	f_{xx}	type
$(0,0)$	-9		saddle point
$(1,1)$	27	6	local min

(c) $\nabla f = \langle 2x, 3y^2 - 3 \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} 2x = 0 \\ 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{matrix} x = 0, \\ y = \pm 1 \end{matrix} \Rightarrow (0,1) \text{ and } (0,-1) \text{ are the critical points}$$

$f_{xx} = 2$, $f_{yy} = 6y$, $f_{xy} = 0 \Rightarrow D(x,y) = 12y$

point	D	f_{xx}	type
$(0,-1)$	-12		saddle point
$(0,1)$	12	2	local min