12.8 Extreme Values (i.e. optimization)

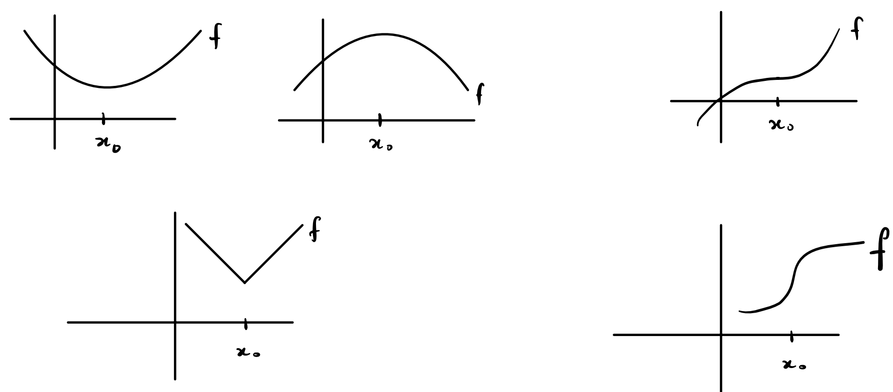
Calc I reminders

A point x_0 is a critical point of f if

$f'(x_0) = 0$ or $f'(x_0)$ does not exist

$f'(x_0) = 0$

$f'(x_0)$ DNE



local extremum at x_0 — no local extremum at x_0

Second derivative test if x_0 is critical point and $f'(x_0) = 0$, then

1. if $f''(x_0) > 0$, local min occurs at x_0
2. if $f''(x_0) < 0$, local max occurs at x_0
3. if $f''(x_0) = 0$, test is inconclusive.

Def Suppose (a,b) is in the domain of f. Then

① f has a local minimum at (a,b) if

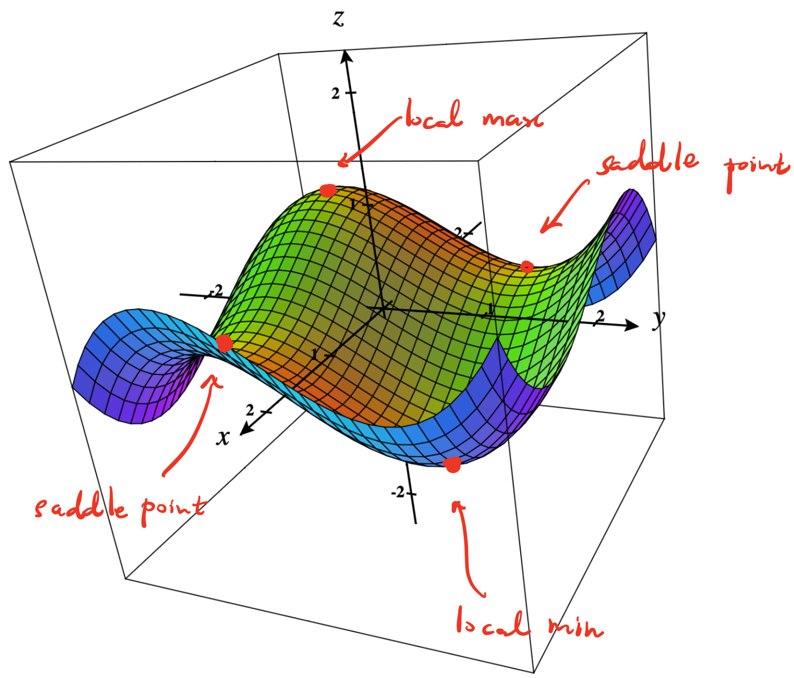
$$f(x,y) \geq f(a,b)$$

for all (x,y) near (a,b)

② f has a local maximum at (a,b) if

$$f(x,y) \leq f(a,b)$$

for all (x,y) near (a,b)

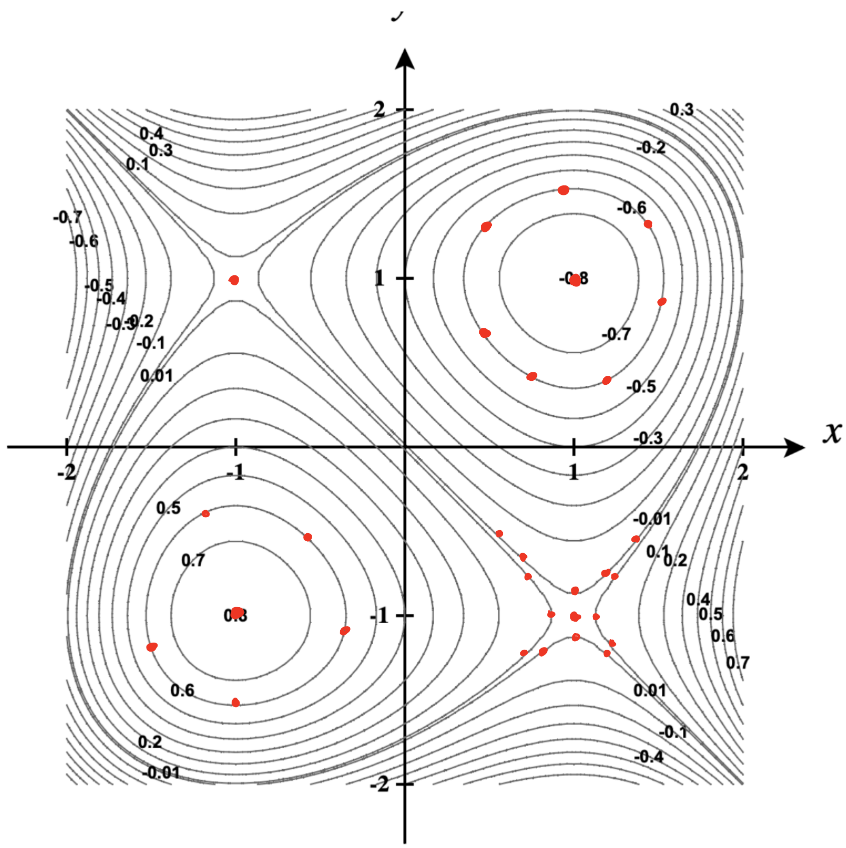


③ f has a saddle point at (a,b) if $\nabla f(a,b) = \vec{0}$

and any region containing (a,b) has a pair of points (x_1, y_1) and (x_2, y_2) such that

$$f(x_1, y_1) < f(a,b) < f(x_2, y_2)$$

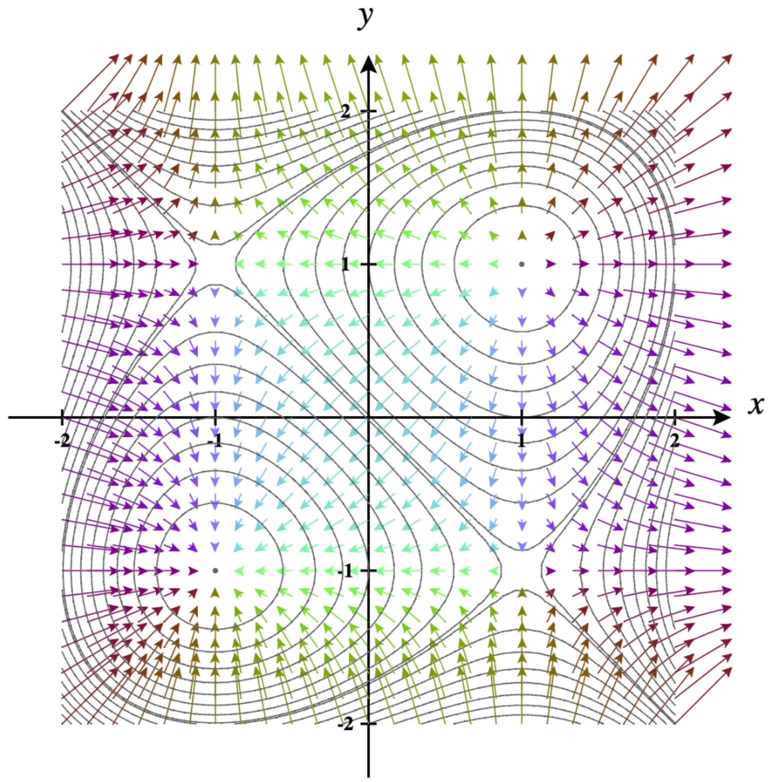
The contour plot of f:



① At each red point P above, which direction does $\nabla f(P)$ point?

② How does the length of $\nabla f(P)$ change as P gets closer to $(1,1)$? $(-1,-1)$? $(1,-1)$? $(-1,1)$?

③ What do you think $\nabla f(1,1)$, $\nabla f(-1,-1)$, $\nabla f(1,-1)$, $\nabla f(-1,-1)$ are?



gradient vector field is shown

Summary The function above has 4 critical points $(\pm 1, \pm 1)$

① Around a local min (like $(1,1)$), $\nabla f(P)$ points away from the local min

② Around a local max (like $(-1,-1)$), $\nabla f(P)$ points toward the local max

③ Around a saddle point (like $(1,-1)$ or $(-1,1)$), $\nabla f(P)$ sometimes points toward, sometimes points away from the saddle

Def A point (x_0, y_0) in the domain of $f(x,y)$ is a critical point of f if $\nabla f(x_0, y_0) = \vec{0}$ or $\nabla f(x_0, y_0)$ is undefined.

Main idea: Critical points are candidates for local extrema. How can we determine if they are local extrema?

Theorem (Second Derivative Test) Let (x_0, y_0) be a critical point with $\nabla f(x_0, y_0) = \vec{0}$ and let

$$D = f_{xx} f_{yy} - f_{xy}^2 \quad (\text{called the discriminant}).$$

Then ① if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local min.

② if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local max.

③ if $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

④ if $D(x_0, y_0) = 0$, then the test is inconclusive.

Remark: $D(x_0, y_0) > 0$ is a condition that $D_{\vec{u}}(D_{\vec{u}} f(x_0, y_0))$ has the same sign (positive or negative) in every direction $\vec{u}$ (guarantees concavity agree in every direction).

Proof of this requires linear algebra.

Example Let $f(x,y) = \frac{1}{5}(x^3+y^3-3x-3y)$. Find all critical points of f and classify them with the Second Derivative Test.

$$\nabla f = \frac{1}{5}\langle 3x^2-3,\ 3y^2-3\rangle$$

$$\nabla f = \vec{0} \quad \text{means} \quad \begin{cases} \frac{1}{5}(3x^2-3)=0 \\ \frac{1}{5}(3y^2-3)=0 \end{cases} \Rightarrow \quad \begin{matrix} x = \pm 1 \\ y = \pm 1 \end{matrix}$$

So $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$ are the critical points of f.

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = 0$, so $D(x,y) = 36xy$

point	D	f_{xx}	type
$(1,1)$	36	6	local min
$(1,-1)$	-36		saddle point
$(-1,1)$	-36		saddle point
$(-1,-1)$	36	-6	local max

Problem 1. True or false?

a. If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, then (a,b) is a critical point of f.

b. The point $(0,0)$ is a critical point of $f(x,y) = \sqrt{x^2+y^2}$.

c. If (a,b) is a critical point of f, then $\nabla f(a,b) = \mathbf{0}$.

d. If f_x and f_y exist at (a,b) and f has a local extremum (ie. local minimum or local maximum) at (a,b) then the tangent plane of f at (a,b) is horizontal (ie. parallel to the xy-plane).

e. If the tangent plane of f at (a,b) is horizontal, then f has a local extremum at (a,b).

(a) true, in this case $\nabla f(a,b) = \langle 0,0 \rangle = \vec{0}$

(b) true :
$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

so $\nabla f(0,0)$ does not exist

(c) false, it might be that $\nabla f(a,b)$ DNE (see above)

(d) true, when $\nabla f(a,b)$ exists and f has a local extremum at (a,b), $\nabla f(a,b) = \langle 0,0 \rangle$.

Therefore the tangent plane at a,b is
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= f(a,b)$$
which is horizontal (planes of the form $z =$ constant are parallel to the xy-plane)

(e) false, if there is a saddle point (ie $(0,0)$ for $f(x,y) = x^2 - y^2$) the tangent plane is horizontal but a saddle is not a local extremum

Problem 1. Find all critical points of the following functions and use the Second Derivative Test to classify them.

a. $f(x,y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$

b. $f(x,y) = 4 + x^3 + y^3 - 3xy$

c. $f(x,y) = x^2 + y^3 - 3y + 1$

(a) $\nabla f = \langle x+4, 4y-8 \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} x+4=0 \\ 4y-8=0 \end{cases} \Rightarrow \begin{matrix} x=-4 \\ y=2 \end{matrix} \Rightarrow (-4,2) \text{ is the only critical point}$$

$f_{xx} = 1$, $f_{yy} = 4$, $f_{xy} = 0 \Rightarrow D(x,y) = 4$

$\Rightarrow D(-4,2) = 4 > 0$, $f_{xx}(-4,2) = 1 > 0$

$\Rightarrow (-4,2)$ is local min

(b) $\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} y = x^2 \\ x = y^2 \end{cases} \Rightarrow x = x^4 \Rightarrow x^4 - x = 0$$

$\Rightarrow x(x^3 - 1) = 0$

$\Rightarrow x = 0, \ x = 1$

So $(0,0)$ and $(1,1)$ are the critical points

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3 \Rightarrow D(x,y) = 36xy - 9$

point	D	f_{xx}	type
$(0,0)$	-9		saddle point
$(1,1)$	27	6	local min

(c) $\nabla f = \langle 2x, 3y^2 - 3 \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} 2x = 0 \\ 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{matrix} x = 0, \\ y = \pm 1 \end{matrix} \Rightarrow (0,1) \text{ and } (0,-1) \text{ are the critical points}$$

$f_{xx} = 2$, $f_{yy} = 6y$, $f_{xy} = 0 \Rightarrow D(x,y) = 12y$

point	D	f_{xx}	type
$(0,-1)$	-12		saddle point
$(0,1)$	12	2	local min