12.8 Extreme Values (Global optimization)

Let f be defined on a set $D \subseteq \mathbb{R}^2$ containing the point (a,b). Then

① f has a global maximum on D at (a,b) if $f(x,y) \le f(a,b)$ for all (x,y) in D.

② f has a global minimum on D at (a,b) if $f(x,y) \ge f(a,b)$ for all (x,y) in D.

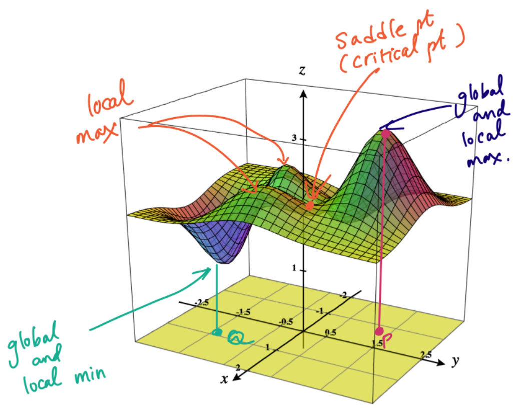


Figure I: $g(x,y)$

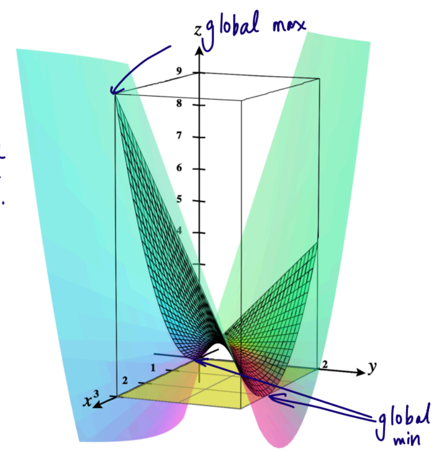


Figure II: $f(x,y)$

$D = \{(x,y) : -2 \le x \le 2,\ -2.5 \le y \le 2.5\}$

$D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$

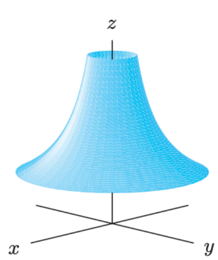


Figure 15.25: Graph showing $f(x,y) = \frac{1}{x^2+y^2}$ has no global maximum on $0 < x^2 + y^2 \le 1$

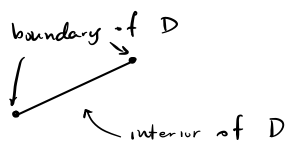
$D = \{(x,y) : 0 < x^2 + y^2 \le 1\}$

- Not every local extremum is a global extremum
- Not every global extremum occurs at a critical point
- Global extrema can occur at multiple points
- Global extrema can fail to exist

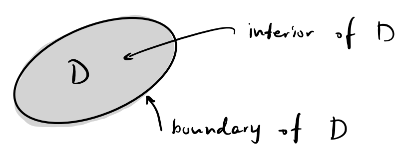
Calc I reminder (Extreme Value Theorem) If $f(x)$ is continuous on $[a,b]$ it has a global max and min which occur at critical points in (a,b) or at endpoints of $[a,b]$.

Calc III version (Extreme Value Theorem) If $f(x,y)$ is continuous on a closed and bounded region $D \subseteq \mathbb{R}^2$, it has a global max and min on D. These occur at critical points of f in the interior of D or at boundary points of D.

Examples (of closed and bounded)



D = line segment

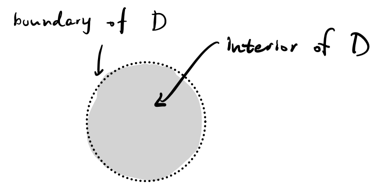


D= filled in 2d region

Non-examples

- $D = \mathbb{R}^2$ not bounded
- $D = \{(x,y) : x^2+y^2 < 1\}$

  not closed.



Finding global extrema on D when f is continuous and D is closed and bounded:

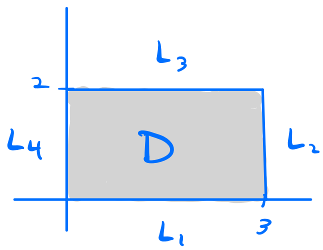
① find critical points of which are contained in the interior of D

② find extrema of f on the boundary of D

③ The largest and smallest of the values found in ① and ② will be the global extrema on D.

Example Let $f(x,y) = x^2 - 2xy + 2y$ and let

$D = \{(x,y) : 0 \le x \le 3, \ 0 \le y \le 2\}$ } figure II above



Step 1 find critical points in the interior of D:

$\nabla f = \langle 2x - 2y, -2x + 2 \rangle$. $\begin{cases} 2x - 2y = 0 \Rightarrow x = y \\ -2x + 2 = 0 \Rightarrow x = 1 \end{cases}$

$(1,1)$ is the only critical point and it's in the interior of D. $\boxed{f(1,1) = 1}$

Step 2 find min, max of f along each boundary segment

(L_1) Notice $y = 0$ and $0 \le x \le 3$, so we want to optimize

$$g_1(x) = f(x,0) = x^2 \quad \text{over } [0,3]$$

$g_1'(x) = 2x$, so $x = 0$ is critical point of g_1

and $\boxed{\begin{array}{l} g_1(0) = f(0,0) = 0 \\ g_1(3) = f(3,0) = 9 \end{array}}$

(L_3) Notice $y = 2$ and $0 \le x \le 3$, so we want to optimize

$$g_3(x) = f(x,2) = x^2 - 4x + 4 \quad \text{over } [0,3]$$

$g_3'(x) = 2x - 4$ so $x = 2$ is critical point of g_3

and

$$g_3(2) = f(2,2) = 0$$
$$g_3(0) = f(0,2) = 4$$
$$g_3(3) = f(3,2) = 1$$

(L_2) Notice $x = 3$ and $0 \le y \le 2$, so we want to optimize

$$g_2(y) = f(3,y) = 9 - 4y \quad \text{over } [0,2]$$

$g_2'(y) = -4$ so g_2 has no critical points

and

$$g_2(0) = f(3,0) = 9$$
$$g_2(2) = f(3,2) = 1$$

(L_4) Notice $x = 0$ and $0 \le y \le 2$, so we want to optimize

$$g_4(y) = f(0,y) = 2y \quad \text{over } [0,2]$$

$g_4'(y) = 2$ so g_4 has no critical points

and

$$g_4(0) = f(0,0) = 0$$
$$g_4(2) = f(0,2) = 4$$

Step 3 Compare all values and conclude:

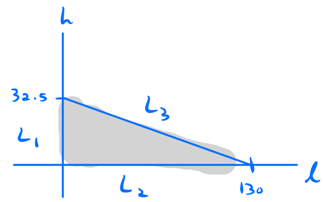
- global max of f is 9, it occurs at $(3,0)$
- global min of f is 0, it occurs at $(0,0)$ and $(2,2)$

Problem 2. The sum of the *length* and *girth* of a rectangular box cannot exceed 130 inches. The *girth* of a box is defined to be twice the sum of its *width* and *height*. Assuming you want to make a box with equal width and height, find the maximum possible volume of such a box under the given constraints.

$\ell + g \leq 130, \quad g = 2(w+h), \quad w = h$

$\Rightarrow \quad \ell + g \leq 130, \quad g = 4h,$

$\Rightarrow \quad \ell + 4h \leq 130, \quad h, \ell > 0$



volume $= f(\ell, h) = h^2 \ell$

$\nabla f = \langle h^2, 2h\ell \rangle, \quad \nabla f = \vec{0}$ when $\begin{cases} h^2 = 0 \\ 2h\ell = 0 \end{cases}$

$\Rightarrow$ points along L_2 are all critical points (where $h = 0$), but this is not a valid dimension for a box (ie. not in the domain of f) so we ignore these.

So max must occur on L_3

Here, $\ell = 130 - 4h$, $0 < h < 32.5$ so we want to optimize

$g(h) = f(130 - 4h, h)$ over $(0, 32.5)$

$= h^2(130 - 4h)$

$= 130h^2 - 4h^3$

$g'(h) = 260h - 12h^2$,

$\Rightarrow \quad h(260 - 12h) = 0$ gives critical points

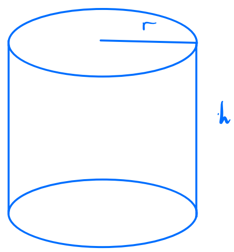
$\Rightarrow \quad h = 0$ (not valid), $h = \frac{260}{12} = \frac{65}{3}$,

$\ell = 130 - 4\left(\frac{65}{3}\right) = \frac{130}{3}$

So max volume is

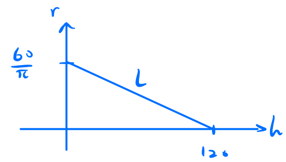
$f\left(\frac{130}{3}, \frac{65}{3}\right) = \boxed{\left(\frac{65}{3}\right)^2 \left(\frac{130}{3}\right)}$

Problem 3. Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 centimeters.



$h + 2\pi r = 120, \quad h, r > 0$

volume $= f(h, r) = \pi r^2 h$



want to maximize f over the line L given by $h + 2\pi r = 120$ where $h, r > 0$.

$$\nabla f = \langle \pi r^2, 2\pi r h \rangle = \vec{0} \Rightarrow \begin{cases} \pi r^2 = 0 \\ 2\pi r h = 0 \end{cases} \Rightarrow \begin{cases} r = 0 \\ r = 0 \text{ or } h = 0 \end{cases}$$

$\Rightarrow$ no critical points on L

We want to maximize

$$g(r) = f(120 - 2\pi r, r) \text{ over } \left(0, \frac{60}{\pi}\right)$$

$$= \pi r^2 (120 - 2\pi r)$$

$$= 120\pi r^2 - 2\pi^2 r^3$$

$$g'(r) = 240\pi r - 6\pi^2 r^2$$

$$= 6\pi r (40 - \pi r)$$

$= 0$ when

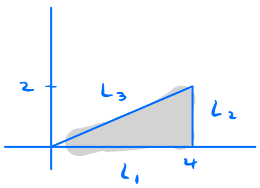
$r = 0$ (not valid), $\quad r = \frac{40}{\pi}, \quad h = 120 - 2\pi\left(\frac{40}{\pi}\right) = 40$

So max volume is

$$f\left(40, \frac{40}{\pi}\right) = \pi \left(\frac{40}{\pi}\right)^2 40$$

$$= \boxed{\frac{40^3}{\pi}}$$

Problem 1. Let $f(x, y) = x^2 - 2xy + 4y^2 - 4x + 24$. Find the global extrema of f on the domain whose boundary is given by the triangle with vertices $(0, 0), (4, 0)$, and $(4, 2)$.



$\nabla f = \langle 2x - 2y - 4,\ 8y - 2x \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} 2x - 2y - 4 = 0 \\ 8y - 2x = 0 \end{cases} \Rightarrow \begin{cases} x = y + 2 \\ x = 4y \end{cases} \Rightarrow 4y = y + 2 \Rightarrow y = \frac{2}{3},\ x = \frac{8}{3}$$

$$f\left(\frac{8}{3}, \frac{2}{3}\right) = \frac{56}{3} \approx 18.67$$

(L_1) $y = 0$, $0 \le x \le 4$, so we want to optimize

$g_1(x) = f(x, 0) = x^2 - 4x + 24$ over $[0, 4]$

$g_1'(x) = 2x - 4 \Rightarrow x = 2$ is critical point

$g_1(0) = 24$
$g_1(2) = 20$
$g_1(4) = 24$

(L_2) $x = 4$, $0 \le y \le 2$, so we want to optimize

$g_2(y) = f(4, y) = 16 - 8y + 4y^2 - 16 + 24$

$= 4y^2 - 8y + 24$ over $[0, 2]$

$g_2'(y) = 8y - 8 \Rightarrow y = 1$ is critical point

$g_2(0) = 24$
$g_2(1) = 20$
$g_2(2) = 24$

(L_3) $y = \frac{1}{2}x$, $0 \le x \le 4$, so we want to optimize

$g_3(x) = f(x, \frac{1}{2}x) = x^2 - 2x\left(\frac{1}{2}x\right) + 4\left(\frac{1}{2}x\right)^2 - 4x + 24$

$= x^2 - 4x + 24$ over $[0, 4]$

we did this already with L_1.

So global min value is 18.67, max value is 24.