Lagrange Multipliers

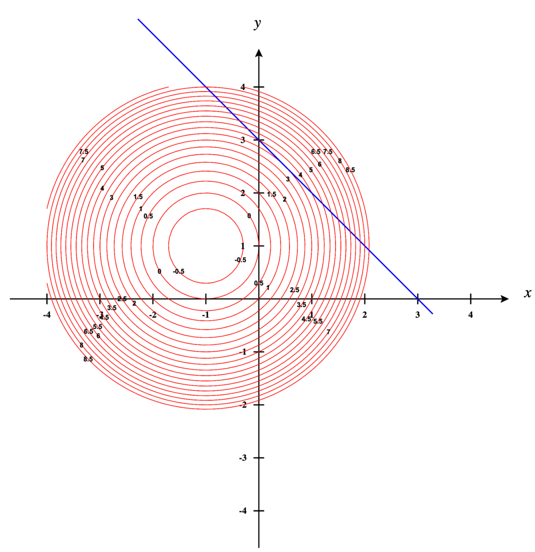
Goal Maximize / minimize a function $f(x,y)$ subject to a constraint.

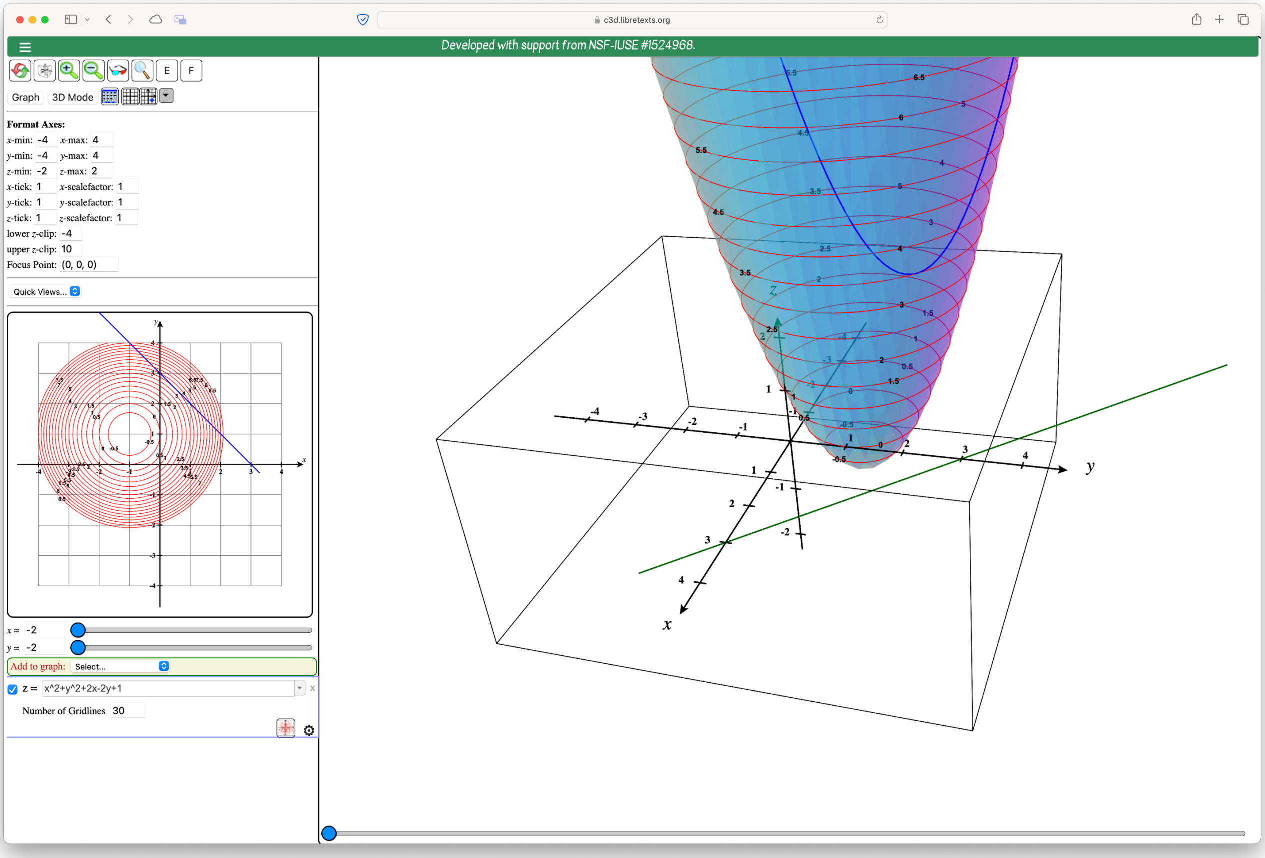
Calc I reminder (Extreme Value Theorem) If $f(x)$ is continuous on $[a,b]$ it has a global max and min which occur at critical points in (a,b) or at endpoints of $[a,b]$.

Warm up Suppose we want to find min / max of $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ subject to the constraint that $\underbrace{x+y}_{g(x,y)} = 3$ and $x, y \geq 0$. The contour plot of f along with the constraint are shown below.

① Where do you think the min / max occur?

② How are ∇f and ∇g related to each other there?





Method of Lagrange Multipliers

To find extreme points (min/max) of $f(x,y)$ subject to the constraint $g(x,y) = c$

① Solve the system

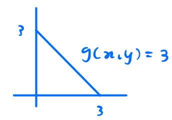
$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = c \end{cases}$$

using λ as a constant (called the Lagrange multiplier of the system).

② Plug in solutions of the system and endpoints of constraint curve into $f(x,y)$ and compare values.

<u>Example</u> Let $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ and $g(x,y) = x + y$.

Find min/max values of f subject to the constraints that $g(x,y) = 3$ and $x, y \geq 0$.



$$\nabla f = \langle 2x+2,\ 2y-2 \rangle, \quad \nabla g = \langle 1, 1 \rangle$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 3 \end{cases} \Rightarrow \begin{cases} 2x + 2 = \lambda \\ 2y - 2 = \lambda \\ x + y = 3 \end{cases} \Rightarrow \begin{cases} 2x + 2 = 2y - 2 \\ x + y = 3 \end{cases}$$

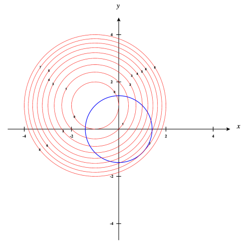
$$\Rightarrow \begin{cases} x - y = -2 \\ x + y = 3 \end{cases} \Rightarrow 2x = 1,\ x = \tfrac{1}{2},\ y = \tfrac{5}{2}$$

So possible locations of min/max are $(\tfrac{1}{2}, \tfrac{5}{2})$, $(3,0)$, and $(0,3)$ (the endpoints)

$f(\tfrac{1}{2}, \tfrac{5}{2}) = 3.5$ (min value)

$f(3,0) = 16$ (max value)

$f(0,3) = 4$



Example find max/min $f(x,y) = x^2 + y^2 + 2x - 2y + 1$

subject to $x^2 + y^2 = 2$

$\nabla f = \langle 2x+2,\ 2y-2 \rangle$

$g(x,y) = x^2 + y^2$

$\nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} 2x + 2 = 2\lambda x & (1) \\ 2y - 2 = 2\lambda y & (2) \\ x^2 + y^2 = 2 & (3) \end{cases}$$

Solve (1),(2) for x, y in terms of λ

and substitute into (3) :

$(1) \implies 2\lambda x - 2x = 2$

$2x(\lambda - 1) = 2$

$x = \dfrac{1}{\lambda - 1}$ (note λ cannot be 1)

$(2) \implies 2\lambda y - 2y = -2$

$2y(\lambda - 1) = -2$

$y = \dfrac{-1}{\lambda - 1}$

$(3) \implies \left(\dfrac{1}{\lambda-1}\right)^2 + \left(\dfrac{-1}{\lambda-1}\right)^2 = 2 \implies \dfrac{2}{(\lambda-1)^2} = 2$

$\implies (\lambda - 1)^2 = 1$

$\implies \lambda - 1 = \pm 1$

$\implies \lambda = 0, 2$

When $\lambda = 0$, $x = -1, y = 1$

When $\lambda = 2$, $x = 1, y = -1$

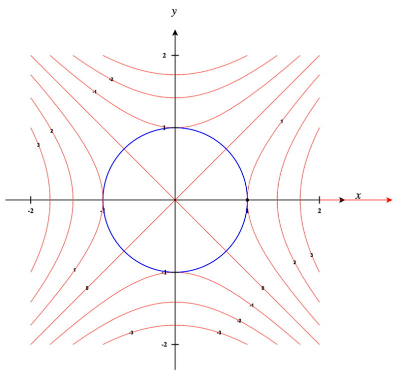
$f(-1,1) = (-1)^2 + (1)^2 + 2(-1) - 2(1) + 1$

$= 1 + 1 - 2 - 2 + 1 = -1$

$f(1,-1) = 1 + 1 + 2 + 2 + 1 = 7$

So minimum occurs at $(-1,1)$, max occurs at $(1,-1)$.

Problem 1. Find the absolute minima and maxima of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$. You might find the contour diagram below helpful in checking your answer.



$\nabla f = \langle 2x, -2y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} 2x = \lambda 2x & (1) \\ -2y = \lambda 2y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

$(1) \Rightarrow 2x - \lambda 2x = 0$

$\Rightarrow 2x(1-\lambda) = 0$

$\Rightarrow x = 0, \lambda = 1$

$(3), x = 0 \Rightarrow 0^2 + y^2 = 1$

$\Rightarrow y = \pm 1$

$(2), \lambda = 1 \Rightarrow -2y = 2y$

$\Rightarrow 4y = 0$

$\Rightarrow y = 0$

$(3) \Rightarrow x^2 + 0^2 = 1$

$\Rightarrow x = \pm 1$

So $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$ are where extreme occur along constraint curve

$f(0,1) = -1$, $f(0,-1) = -1$ give absolute min

$f(1,0) = 1$, $f(-1,0) = 1$ give absolute max

Problem 2. A manufacturer of golf balls has created a model (ie. a function)

$$f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$$

which outputs the profit of monthly sales (in thousands of dollars), given x golf balls sold per month (in thousands) and y hours per month of advertising. Every thousand golf balls cost \$20 (thousand dollars) to produce and every hour of advertising costs \$4 (thousand dollars). Find values of x and y that maximize profit subject to the constraint that there is a fixed budget of \$216 thousand dollars.

Constraint: $\underbrace{20x + 4y}_{= g(x,y)} = 216$

$\nabla f = \langle 48 - 2x - 2y, 96 - 2x - 18y \rangle$

$\nabla g = \langle 20, 4 \rangle$

$$\begin{cases} 48 - 2x - 2y = 20\lambda & (1) \\ 96 - 2x - 18y = 4\lambda & (2) \\ 20x + 4y = 216 & (3) \end{cases}$$

$(1) = 5(2) \Rightarrow 48 - 2x - 2y = 5(96 - 2x - 18y)$

$\Rightarrow 8x + 88y = 432 \Rightarrow 88y = 432 - 8x$

$(3) \Rightarrow 22(20x + 4y) = 22(216)$

$\Rightarrow 440x + 88y = 4752$

$\Rightarrow 440x + 432 - 8x = 4752$

$\Rightarrow 432x = 4320$

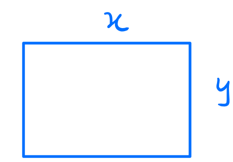
$\Rightarrow x = 10$

$\Rightarrow 88y = 432 - 8(10)$

$y = 4$

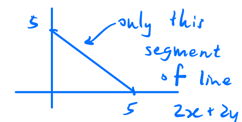
$\boxed{\text{So } (10,4) \text{ maximize profit and the max value is } f(10,4) = 540 \quad (\text{min occurs when } x=0,\ y=54 \text{ or } y=0,\ x=10.8)}$

Problem 3. Find the dimensions of a rectangular region with largest area, subject to the constraint that its perimeter be 10 meters.



optimize $f(x,y) = xy$, subject to $\underbrace{2x+2y=10}_{g(x,y) = 2x+2y}$

Constraint curve



$2x+2y=10$, since $x, y \geq 0$.

$\nabla f = \langle y, x \rangle$,

$\nabla g = \langle 2, 2 \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 10 \end{cases} \Rightarrow \begin{cases} y = 2\lambda \\ x = 2\lambda \\ 2x + 2y = 10 \end{cases}$$

$$\Rightarrow \quad 2(2\lambda) + 2(2\lambda) = 10$$

$$8\lambda = 10, \quad \lambda = 5/4$$

So optimal dimensions are $x = \frac{5}{2}$, $y = \frac{5}{2}$

$f(5/4, 5/4) = \frac{25}{9}$ is maximum area,

endpoints $(5,0)$, $(0,5)$ give min area of $f(5,0) = f(0,5) = 0$ (a "rectangle" where one dimension is 0 has 0 area)