

Method of Lagrange Multipliers

Summary At the points (a, b) where $f(x, y)$

has a max/min subject to the constraint

$g(x, y) = C$, the level curve of f through (a, b)

is tangent to $g(x, y) = C$. This means $\nabla f(a, b)$ and

$\nabla g(a, b)$ are parallel: $\nabla f(a, b) = \lambda \nabla g(a, b)$

for some scalar λ called a Lagrange multiplier.

(Joseph-Louis Lagrange, 1736-1813: mathematician and astronomer)

Method of Lagrange Multipliers To find extrema

of $f(x, y)$ subject to the constraint $g(x, y) = C$

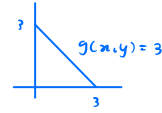
① Solve the system

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = C \end{cases}$$

② Plug in solutions of ① and any endpoints of the constraint into $f(x, y)$ and compare values.

Example Let $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ and $g(x,y) = x + y$.

Find min/max values of f subject to the constraints that $g(x,y) = 3$ and $x, y \geq 0$.



$$\nabla f = \langle 2x+2, 2y-2 \rangle, \quad \nabla g = \langle 1, 1 \rangle$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 3 \end{cases} \Rightarrow \begin{cases} 2x+2 = \lambda \\ 2y-2 = \lambda \\ x+y = 3 \end{cases} \Rightarrow \begin{cases} 2x+2 = 2y-2 \\ x+y = 3 \end{cases}$$
$$\Rightarrow \begin{cases} x-y = -2 \\ x+y = 3 \end{cases} \Rightarrow 2x = 1, x = \frac{1}{2}, y = \frac{5}{2}$$

So possible locations of min/max are $(\frac{1}{2}, \frac{5}{2})$,

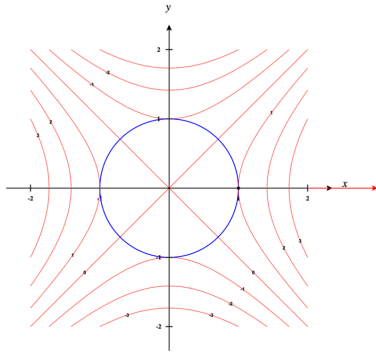
$(3, 0)$, and $(0, 3)$ (the endpoints)

$$f(\frac{1}{2}, \frac{5}{2}) = 3.5 \quad (\text{min value})$$

$$f(3, 0) = 16 \quad (\text{max value})$$

$$f(0, 3) = 4$$

Problem 1. Find the absolute minima and maxima of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$. You might find the contour diagram below helpful in checking your answer.



$$\begin{aligned} \nabla f &= \langle 2x, -2y \rangle \\ \nabla g &= \langle 2x, 2y \rangle \end{aligned} \quad \left\{ \begin{array}{l} 2x = \lambda 2x \quad (1) \\ -2y = \lambda 2y \quad (2) \\ x^2 + y^2 = 1 \quad (3) \end{array} \right.$$

$$\begin{aligned} (1) \Rightarrow 2x - \lambda 2x &= 0 \\ \Rightarrow 2x(1 - \lambda) &= 0 \\ \Rightarrow x = 0, \lambda = 1 \end{aligned}$$

$$\begin{aligned} (3), x=0 \Rightarrow 0^2 + y^2 &= 1 \\ \Rightarrow y &= \pm 1 \end{aligned} \quad \begin{aligned} (2), \lambda=1 \Rightarrow -2y &= 2y \\ \Rightarrow 4y &= 0 \\ \Rightarrow y &= 0 \end{aligned}$$

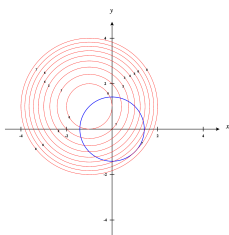
$$\begin{aligned} \text{So } (0, 1), (0, -1), \\ (1, 0), (-1, 0) \text{ are} \end{aligned}$$

$$\begin{aligned} (3) \Rightarrow x^2 + 0^2 &= 1 \\ \Rightarrow x &= \pm 1 \end{aligned}$$

where extrema occur along constraint curve

$$f(0, 1) = -1, f(0, -1) = -1 \text{ give absolute min}$$

$$f(1, 0) = 1, f(-1, 0) = 1 \text{ give absolute max}$$



Problem 2. Find the global extrema of $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ subject to the constraint $x^2 + y^2 = 2$.

$$\begin{aligned} \nabla f &= \langle 2x+2, 2y-2 \rangle \\ g(x,y) &= x^2 + y^2 \\ \nabla g &= \langle 2x, 2y \rangle \end{aligned} \quad \left\{ \begin{array}{l} 2x+2 = 2\lambda x \quad (1) \\ 2y-2 = 2\lambda y \quad (2) \\ x^2 + y^2 = 2 \quad (3) \end{array} \right.$$

Solve (1),(2) for x,y in terms of λ
and substitute into (3):

$$\begin{aligned} (1) \Rightarrow 2\lambda x - 2x &= 2 \\ 2x(\lambda - 1) &= 2 \\ x &= \frac{1}{\lambda - 1} \quad (\text{note } \lambda \text{ cannot be } 1) \end{aligned}$$

$$\begin{aligned} (2) \Rightarrow 2\lambda y - 2y &= -2 \\ 2y(\lambda - 1) &= -2 \\ y &= \frac{-1}{\lambda - 1} \end{aligned}$$

$$\begin{aligned} (3) \Rightarrow \left(\frac{1}{\lambda - 1}\right)^2 + \left(\frac{-1}{\lambda - 1}\right)^2 &= 2 \Rightarrow \frac{2}{(\lambda - 1)^2} = 2 \\ &\Rightarrow (\lambda - 1)^2 = 1 \\ &\Rightarrow \lambda - 1 = \pm 1 \\ &\Rightarrow \lambda = 0, 2 \end{aligned}$$

When $\lambda = 0$, $x = -1, y = 1$

When $\lambda = 2$, $x = 1, y = -1$

$$\begin{aligned} f(-1, 1) &= (-1)^2 + (1)^2 + 2(-1) - 2(1) + 1 \\ &= 1 + 1 - 2 - 2 + 1 = -1 \end{aligned}$$

$$f(1, -1) = 1 + 1 + 2 + 2 + 1 = 7$$

So minimum occurs at $(-1, 1)$, max occurs at $(1, -1)$.

Problem 3. A manufacturer of golf balls has created a model (ie. a function)

$$f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$$

which outputs the profit of monthly sales (in thousands of dollars), given x golf balls sold per month (in thousands) and y hours per month of advertising. Every thousand golf balls cost \$20 (thousand dollars) to produce and every hour of advertising costs \$4 (thousand dollars). Find values of x and y that maximize profit subject to the constraint that there is a fixed budget of \$216 thousand dollars.

$$\text{Constraint: } \underbrace{20x + 4y}_{=g(x,y)} = 216$$

$$\nabla f = \langle 48 - 2x - 2y, 96 - 2x - 18y \rangle$$

$$\nabla g = \langle 20, 4 \rangle$$

$$\begin{cases} 48 - 2x - 2y = 20\lambda & \textcircled{1} \\ 96 - 2x - 18y = 4\lambda & \textcircled{2} \\ 20x + 4y = 216 & \textcircled{3} \end{cases}$$

$$\textcircled{1} = 5\textcircled{2} \Rightarrow 48 - 2x - 2y = 5(96 - 2x - 18y)$$

$$\Rightarrow 8x + 88y = 432 \Rightarrow 88y = 432 - 8x$$

$$\textcircled{3} \Rightarrow 22(20x + 4y) = 22(216)$$

$$\Rightarrow 440x + 88y = 4752$$

$$\Rightarrow 440x + 432 - 8x = 4752$$

$$\Rightarrow 432x = 4320$$

$$\Rightarrow x = 10$$

$$\Rightarrow 88y = 432 - 8(10)$$

$$y = 4$$

So $(10, 4)$ maximize profit and the max value is $f(10, 4) = 540$
(min occurs when $x=0, y=54$ or $y=0, x=10.8$)

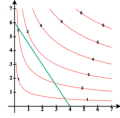
Problem 4. An economic model for consumer behavior tries to capture how much happiness (or *utility*) an individual derives from combinations of leisure and goods. In this model, utility U is given as a function $f(x, y)$ of the amount of leisure x and goods y the individual has. We suppose

$$U = f(x, y) = x^{1/3} y^{2/3}$$

and we suppose that x and y are non-negative quantities. Since life has constraints, it is natural to assume that the leisure and goods an individual can have by the equation

$$3x + 2y = 12.$$

Find the optimal combination of leisure and goods that maximizes utility.



$$f(x, y) = x^{1/3} y^{2/3} \quad \nabla f = \left\langle \frac{1}{3} x^{-2/3} y^{2/3}, \frac{2}{3} x^{1/3} y^{-1/3} \right\rangle$$

$$g(x, y) = 3x + 2y \quad \nabla g = \langle 3, 2 \rangle$$

$$\begin{cases} \frac{1}{3} x^{-2/3} y^{2/3} = 3\lambda & (1) \\ \frac{2}{3} x^{1/3} y^{-1/3} = 2\lambda & (2) \\ 3x + 2y = 12 & (3) \end{cases}$$

$$(1) \Rightarrow \frac{1}{9} \left(\frac{x}{y} \right)^{-2/3} = \lambda$$

$$(2) \Rightarrow \frac{1}{3} \left(\frac{x}{y} \right)^{1/3} = \lambda$$

$$\Rightarrow \frac{1}{9} \left(\frac{x}{y} \right)^{-2/3} = \frac{1}{3} \left(\frac{x}{y} \right)^{1/3}$$

$$\Rightarrow \frac{1}{3} = \frac{x}{y}$$

$$\Rightarrow y = 3x$$

$$(3) \Rightarrow 3x + 2(3x) = 12$$

$$\Rightarrow 9x = 12$$

$$\Rightarrow x = 4/3, y = 4$$

$$f(4/3, 4) = (4/3)^{1/3} (4)^{2/3}$$

↑
this combination maximizes utility

end
points
of
constraint

$$\left\{ \begin{array}{l} f(0, 6) = 0 \\ f(4, 0) = 0 \end{array} \right.$$

