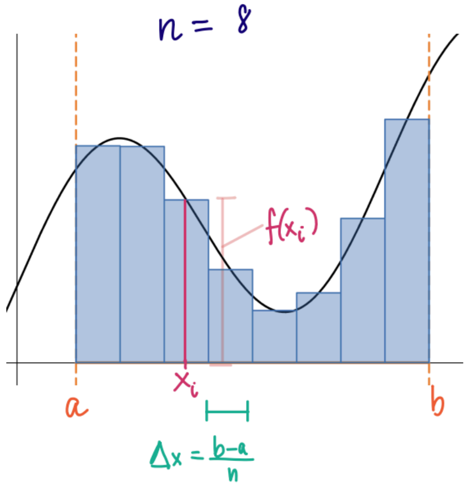
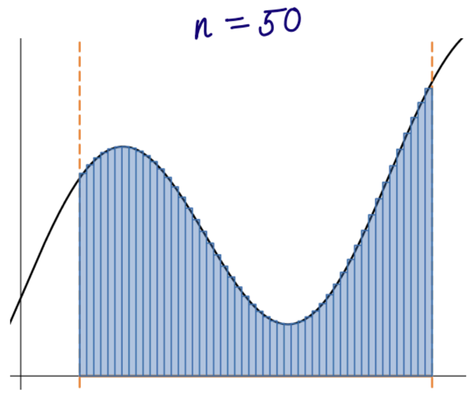
§ 13.1 Double Integrals (over rectangles)





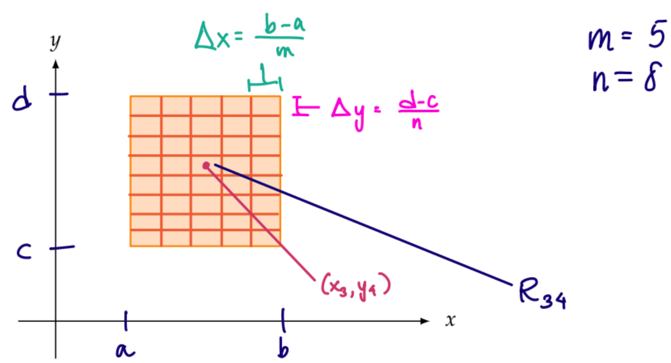
In calculus I and II we introduce the definite integral as a way to find the signed area under $f(x)$ over a given interval $[a,b]$.

Divide $[a,b]$ into n subintervals each of length $\Delta x = \frac{b-a}{n}$ and choose representative points $x_1, \dots, x_n$ from each subinterval.

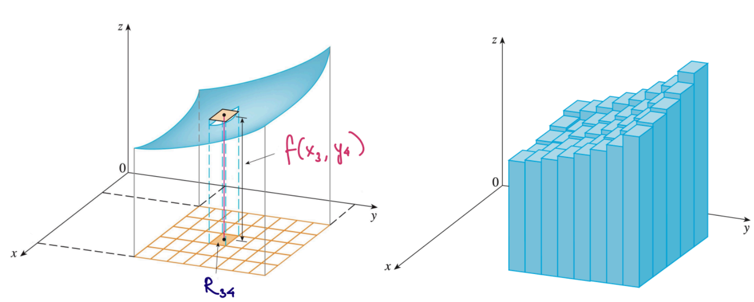
$$\int_a^b f(x)\,dx = \lim_{\substack{\Delta x \to 0 \\ \text{so } n \to \infty}} \overbrace{\sum_{i=1}^{n} \underbrace{f(x_i)\,\Delta x}_{\text{area of } i\text{th rectangle}}}^{\text{Riemann sum}}$$

Today, we'll define a double integral of a function $f(x,y)$ as a <u>signed</u> <u>volume</u> over a region R in the xy-plane. For now we'll suppose R is a rectangle:

$$R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$



We'll subdivide R into mn subrectangles R_{ij} where $1 \le i \le m$ and $1 \le j \le n$. Each will have area $\Delta A = \Delta x \Delta y$



For each rectangle R_{ij} we choose an arbitrary point (x_i, y_j) and the volume of the rectangular prism with height $f(x_i, y_j)$ is

$$f(x_i, y_j)\,\Delta A = f(x_i, y_j)\,\Delta x\,\Delta y.$$

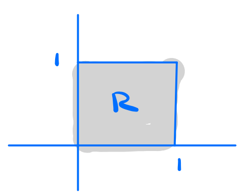
And so we define the double integral as:

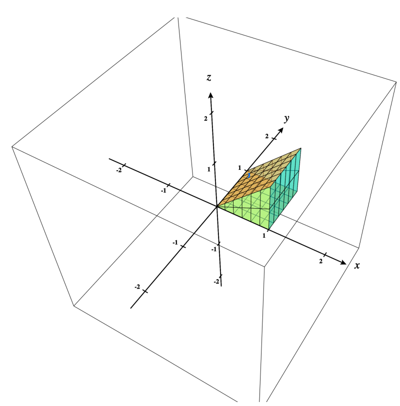
$$\iint_R f(x,y)\,dA = \lim_{\substack{\Delta x \to 0,\\ \Delta y \to 0}} \sum_{j=1}^{n} \sum_{i=1}^{m} \underbrace{f(x_i, y_j)\Delta x \Delta y}_{\text{volume of } i,j^{\text{th}} \text{ rectangular prism}}$$

so $n, m \to \infty$

Example Find the sign (positive, negative, or zero) of

(1) $\displaystyle\iint_R x\,dA$ where $R = [0,1]\times[0,1]$



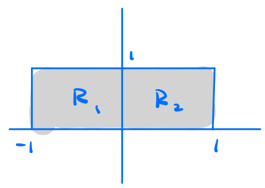


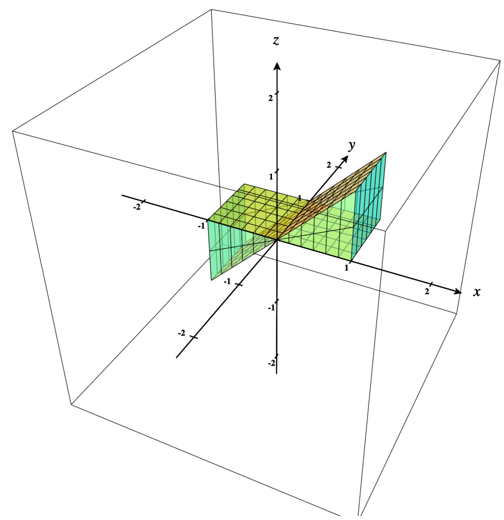
$f(x,y) = x$ is non-negative for all (x,y) on R

Therefore integral is positive

$f(x,y) = x$ is slanted plane

(2) $\displaystyle\iint_R x\,dA$ where $R = [-1,1]\times[0,1]$





$R = R_1 \cup R_2$ $\quad f(x,y) = x$ is positive on R_2 and negative on R_1

By symmetry, $\displaystyle\iint_{R_1} x\,dA = -\iint_{R_2} x\,dA$

And $\displaystyle\iint_R x\,dA = 0$

To actually compute $\displaystyle\iint_R f(x,y)\,dA$ we do what is called an iterated integral. When $R = [a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx \right)}_{\substack{\text{treat } y \text{ like a constant} \\ \text{and integrate wrt } x}} dy$$

$$\text{or} \quad = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy \right)}_{\substack{\text{treat } x \text{ like a constant} \\ \text{and integrate wrt } y}} dx$$

Example Compute $\iint_R f(x,y)\,dA$ where $f(x,y) = x+y^2$,

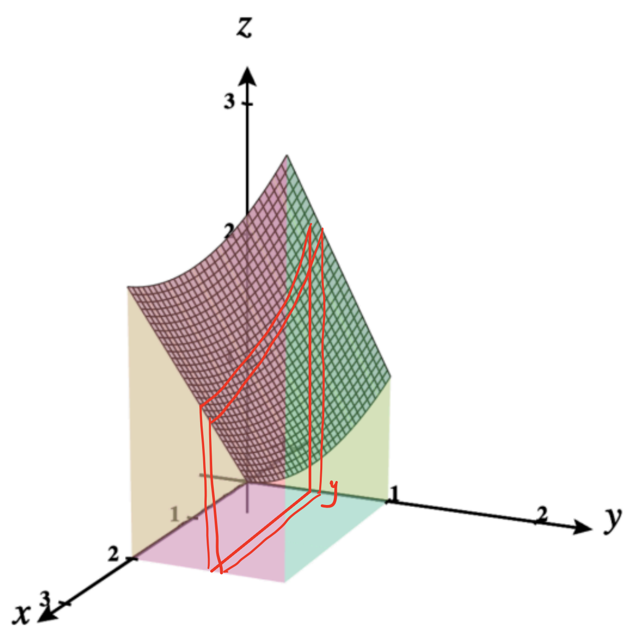
$R = [0,2] \times [0,1]$ and $dA = dx\,dy$

$$\int_0^1 \underbrace{\left(\int_0^2 (x+y^2)\,dx\right)}_{\text{volume of a slice for an arbitrary choice of } y} dy = \int_0^1 \left(\tfrac{1}{2}x^2 + xy^2 \Big|_0^2 \right) dy$$

$$= \int_0^1 \left(2 + 2y^2 \right) dy$$

$$= 2y + \tfrac{2}{3}y^3 \Big|_0^1$$

$$= 2 + \tfrac{2}{3} = \tfrac{8}{3}$$



volume of this slice is

$$\int_0^2 (x+y^2)\,dx = 2 + 2y^2$$

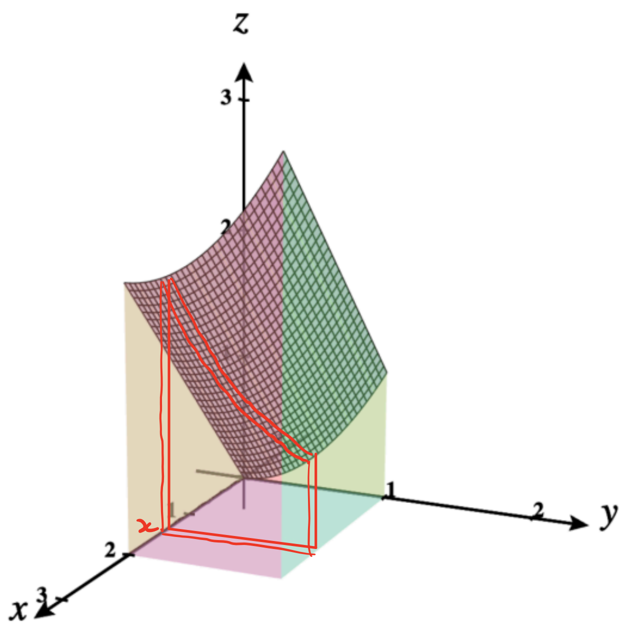
Remark We'd get the same answer using $dA = dy\,dx$.

$$\int_0^2 \underbrace{\left(\int_0^1 (x+y^2)\,dy\right)}_{\text{volume of a slice for an arbitrary choice of } x} dx = \int_0^2 \left(xy + \tfrac{1}{3}y^3 \Big|_0^1 \right) dx$$

$$= \int_0^2 \left(x + \tfrac{1}{3} \right) dx$$

$$= \tfrac{1}{2}x^2 + \tfrac{1}{3}x \Big|_0^2$$

$$= 2 + \tfrac{2}{3} = \tfrac{8}{3}.$$



volume of this slice is

$$\int_0^1 (x+y^2)\,dy = x + \tfrac{1}{3}$$

Example Compute $\iint_R x e^{xy}\, dA$ where $R = [0, \ln 2] \times [0, 1]$

using either $dA = dx\,dy$ or $dA = dy\,dx$, whichever is easier.

dxdy : $\underbrace{\int_0^1 \int_0^{\ln 2} x e^{xy}\, dx\, dy}_{\text{requires integration by parts!}}$

dydx : $\int_0^{\ln 2} \int_0^1 x e^{xy}\, dy\, dx$

remember $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$

$$= \int_0^{\ln 2} x \left(\int_0^1 e^{xy}\, dy \right) dx$$

$$= \int_0^{\ln 2} x \left(\frac{1}{x} e^{xy} \Big|_0^1 \right) dx$$

$$= \int_0^{\ln 2} (e^x - 1)\, dx$$

$$= e^x - x \Big|_0^{\ln 2}$$

$$= 2 - \ln 2 - 1$$

$$= 1 - \ln 2$$

Problem 1. Compute the following double integrals.

a. $\int_0^3 \int_0^1 (5x^3 + 3xy - 4y^2)\, dy dx$

b. $\int_0^\pi \int_0^{\pi/2} (x^2 \sin y + \cos x)\, dx dy$

c. $\int_1^2 \int_0^2 (y + x/y^2)\, dx dy$

d. $\int_0^e \int_1^e 2x/y\, dy dx$

(a) $$\int_0^3 \left(5x^3 y + \frac{3}{2} xy^2 - \frac{4}{3} y^3 \Big|_0^1 \right) dx$$

$$= \int_0^3 \left(5x^3 + \frac{3}{2} x - \frac{4}{3} \right) dx$$

$$= \frac{5}{4} x^4 + \frac{3}{4} x^2 - \frac{4}{3} x \Big|_0^3$$

$$= \frac{5}{4}(3)^4 + \frac{3}{4}(3)^2 - 4$$

$$= 104$$

(b) $$\int_0^\pi \left(\frac{1}{3} x^3 \sin y + \sin x \Big|_0^{\pi/2} \right) dy$$

$$= \int_0^\pi \left(\frac{\pi^3}{24} \sin y + 1 \right) dy$$

$$= -\frac{\pi^3}{24} \cos y + y \Big|_0^\pi$$

$$= \frac{\pi^3}{12} + \pi$$

(c) $$\int_1^2 \left(xy + \frac{x^2}{2y^2} \Big|_0^2 \right) dy$$

$$= \int_1^2 \left(2y + \frac{2}{y^2} \right) dy$$

$$= y^2 - 2y^{-1} \Big|_1^2$$

$$= (4 - 1) - (1 - 2)$$

$$= 4$$

(d) $$\int_0^e \left(2x \ln|y| \Big|_1^e \right) dx$$

$$= \int_0^e 2x \, dx$$

$$= x^2 \Big|_0^e$$

$$= e^2$$