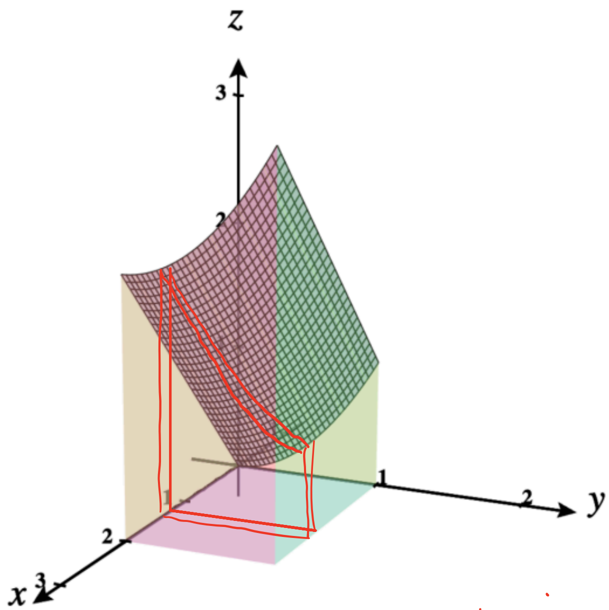
§ 13.1, 13.2 Double Integrals (over non-rectangular regions)

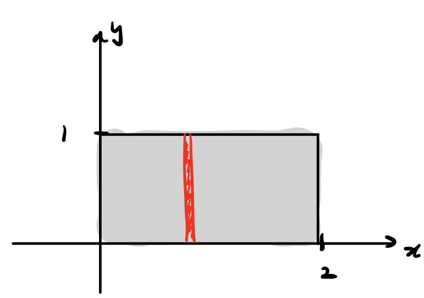
Last time we introduced $\iint_R f(x,y)\,dA$ when $R = [a,b] \times [c,d]$. Today we'll discuss how to approach double integrals for other regions.

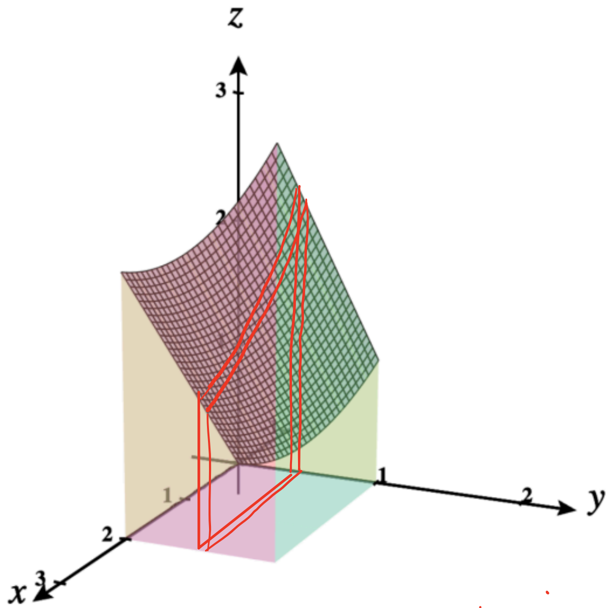


$$\int_{x=0}^{x=2} \Bigg(\underbrace{\int_{y=0}^{y=1} f(x,y)\, dy}_{} \Bigg)\, dx$$

volume of
slice parallel to y-axis
for fixed arbitrary x

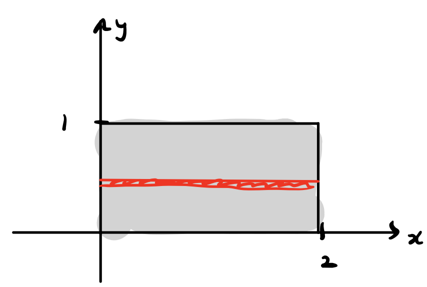
Reminder when $dA = dy\,dx$ we sliced
parallel to y-axis





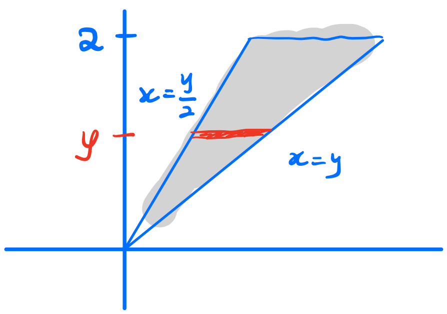
$$\int_{y=0}^{y=1}\left(\underbrace{\int_{x=0}^{x=2} f(x,y)\,dx}_{\substack{\text{volume of} \\ \text{slice parallel to } x\text{-axis} \\ \text{for fixed arbitrary } y}}\right)dy$$

Reminder when $dA = dx\,dy$ we sliced parallel to x-axis

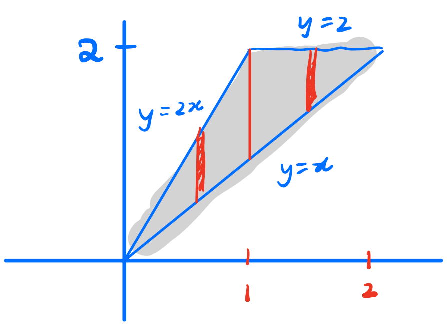


Example Let R be the region between $y=x$ and $y=2x$ for y-values $0 \le y \le 2$. Sketch R and set up

$\iint_R f(x,y)\,dA$ using ① $dA = dx\,dy$ ② $dA = dy\,dx$.



$$\underbrace{\int_{y=0}^{y=2}\left(\underbrace{\int_{x=y/2}^{x=y} f(x,y)\,dx}_{\text{fix } y\text{, slice parallel to } x\text{-axis.}}\right)dy}_{\text{then sum slice volumes from } y=0 \text{ to } y=2}$$



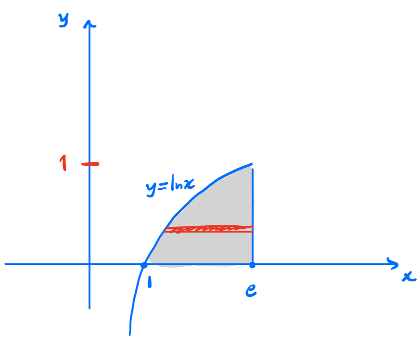
$$\underbrace{\int_{x=0}^{x=1}\left(\underbrace{\int_{y=x}^{y=2x} f(x,y)\,dy}_{\text{fix } x\text{, slice parallel to } y\text{-axis.}}\right)dx}_{\text{then sum slice volumes from } x=0 \text{ to } x=1.} + \underbrace{\int_1^2 \underbrace{\int_{y=x}^{y=2} f(x,y)\,dy}_{\text{fix } x\text{, slice parallel to } y\text{-axis.}}\,dx}_{\text{then sum slice volumes from } x=1 \text{ to } x=2}$$

Rule of thumb Inside limits of integration can involve a variable, but outside limits cannot.

Example Sketch the region of integration of

$$\int_1^e \int_0^{\ln x} 1 \, dy \, dx$$

Then set up the integral with $dA = dx\,dy$ and evaluate it.



$$\int_{y=0}^{y=1} \int_{x=e^y}^{x=e} 1 \, dx\, dy = \int_0^1 \left(x \Big|_{x=e^y}^{x=e} \right) dy$$

$$= \int_0^1 (e - e^y)\, dy$$

$$= ey - e^y \Big|_0^1$$

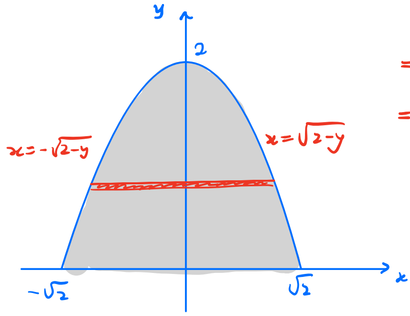
$$= (e - e) - (0 - e^0)$$

$$= 1$$

Example Sketch the region of integration of

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{2-x^2} 1 \, dy\, dx$$

Then set up the integral with $dA = dx\,dy$.



$y = 2 - x^2$

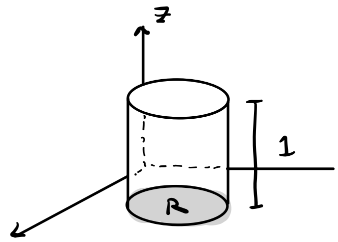
$\Rightarrow \; x^2 = 2 - y$

$\Rightarrow \; x = \pm\sqrt{2-y}$

$$\int_{y=0}^{y=2} \int_{x=-\sqrt{2-y}}^{x=\sqrt{2-y}} 1 \, dx\, dy$$

Remark When $f(x,y) = 1$, the integral $\iint_R 1\, dA$

is equal to area$(R) \cdot 1 =$ area(R).

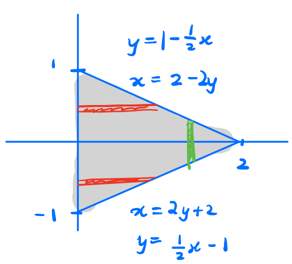


Example Let R be the region bounded by

$$x = 0, \quad x + 2y = 2, \quad x - 2y = 2.$$

Set up $\iint_R f(x,y)\, dA$ using ① $dA = dy\,dx$.

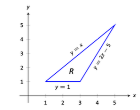
and ② $dA = dx\,dy$

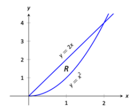


① $$\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x-1}^{y=1-\frac{1}{2}x} f(x,y)\, dy\,dx$$

② $$\int_{y=-1}^{y=0} \int_{x=0}^{x=2y+2} f(x,y)\, dx\,dy + \int_{y=0}^{y=1} \int_{x=0}^{x=2-2y} f(x,y)\, dx\,dy$$

Problem 1. Let $f(x,y)$ be a given function. For each region R below, set up the double integral $\iint_R f(x,y)\,dA$ in two ways: using $dA = dydx$ and $dA = dxdy$.





(a) $$\int_{x=1}^{x=3}\int_{y=1}^{y=x} f(x,y)\,dydx + \int_{x=3}^{x=5}\int_{y=2x-5}^{y=x} f(x,y)\,dydx$$

$$\int_{y=1}^{y=5}\int_{x=y}^{x=\frac{1}{2}(y+5)} f(x,y)\,dxdy$$

(b) $$\int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx$$

$$\int_{y=0}^{y=4}\int_{x=\frac{1}{2}y}^{x=\sqrt{y}} f(x,y)\,dxdy$$

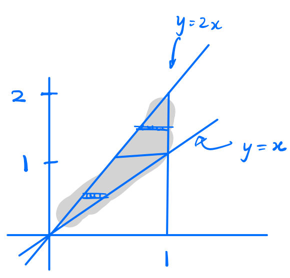
Problem 2. Each double integral below represents the area of a region R in the xy-plane. Sketch R and then set up the integral again with the order of integration reversed.

a. $\int_0^1 \int_x^{2x} 1\,dydx$

b. $\int_0^{10} \int_{-10+y}^{10-y} 1\,dxdy$

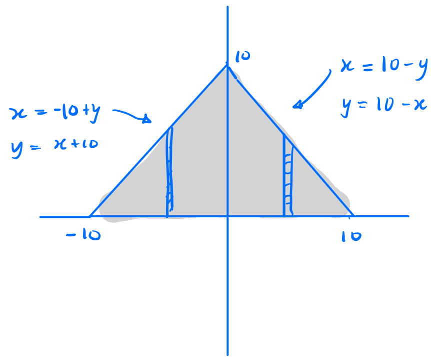
c. $\int_{-1}^{1} \int_{y^2}^{1} 1\,dxdy$

(a)



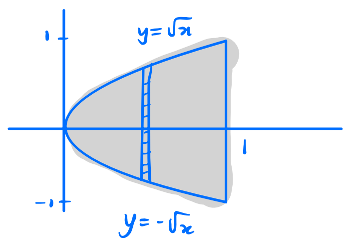
$$\int_{y=0}^{y=1}\int_{x=y/2}^{x=y} 1dxdy + \int_{y=1}^{y=2}\int_{x=y/2}^{x=1} 1dxdy$$

(b)



$$\int_{x=-10}^{x=0}\int_{y=0}^{y=x+10} 1\,dy\,dx + \int_{x=0}^{x=10}\int_{y=0}^{10-x} 1\,dy\,dx$$

(c)



$$\int_{x=0}^{x=1}\int_{y=-\sqrt{x}}^{y=\sqrt{x}} 1\,dy\,dx$$

Problem 3. Let D be the region inside the unit circle in $\mathbb{R}^2$, let R be the right half of D, and let B be the bottom half of D. Determine whether each of the following double integrals is positive, negative, or zero by thinking about the (signed) volume that each represents and whether the function in the integrand is positive, negative, or zero over the region of integration. You might find it helpful to use CalcPlot3d.

a. $\iint_D 1\,dA$

b. $\iint_D (1-\sqrt{x^2+y^2})\,dA$

c. $\iint_D (-1+x^2+y^2)\,dA$

d. $\iint_R x\,dA$

e. $\iint_R y\,dA$

f. $\iint_B x\,dA$

g. $\iint_B y\,dA$

part	sign
a.	+
b.	+
c.	−
d.	+
e.	0
f.	0
g.	−