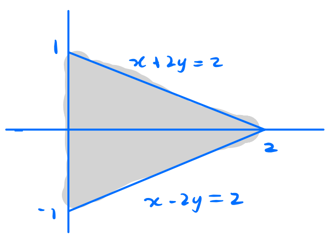
13.2 More Double Integrals

Goal get more practice setting up double integrals and look at examples where it's necessary to change the order of integration.

Example Let $f(x,y) = xy$ and suppose R is the region in $\mathbb{R}^2$ bounded by $x=0$, $x+2y=2$, and $x-2y=2$. Set up $\iint_R f(x,y)\,dA$ with both orders of integration and compute one of them.



① $$\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x-1}^{y=1-\frac{1}{2}x} f(x,y)\,dy\,dx$$

② $$\int_{y=-1}^{y=0} \int_{x=0}^{x=2+2y} f(x,y)\,dx\,dy + \int_{y=0}^{y=1} \int_{x=0}^{x=2-2y} f(x,y)\,dx\,dy$$

We'll compute ①:

$$\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x-1}^{y=1-\frac{1}{2}x} xy\,dy\,dx$$

$$= \int_{x=0}^{x=2} x\left(\frac{1}{2}y^2\Big|_{y=\frac{1}{2}x-1}^{y=1-\frac{1}{2}x}\right)dx$$

$$= \int_0^2 \frac{1}{2}x\left(\left(1-\frac{1}{2}x\right)^2 - \left(\frac{1}{2}x-1\right)^2\right)dx$$

$$= \int_0^2 \frac{1}{2}x\left(\left(1-x+\frac{1}{4}x^2\right) - \left(\frac{1}{4}x^2-x+1\right)\right)dx$$

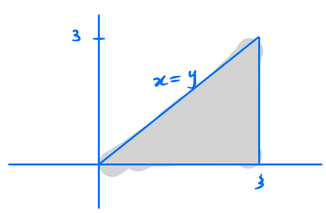
$= 0$ symmetry!

Example Compute the integral $\int_0^3 \int_y^3 e^{-x^2}\, dx\, dy$.

Issue It's not possible to find a formula for the antiderivative of e^{-x^2} with respect to x.

We're going to compute this integral by changing the order of integration so that we integrate with respect to y on the inside integral.

Warning You have to draw a picture of R when changing order of integration. $\int_y^3 \int_0^3$ is not valid.



$$\int_{x=0}^{x=3} \int_{y=0}^{y=x} e^{-x^2}\, dy\, dx$$

$$= \int_{x=0}^{x=3} e^{-x^2} \left(y \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_{x=0}^{x=3} x e^{-x^2}\, dx \qquad u = -x^2$$

$$du = -2x\,dx$$

$$= \int_0^{-9} -\tfrac{1}{2} e^u\, du \qquad -\tfrac{1}{2} du = x\,dx$$

$$= \frac{1}{2} \int_{-9}^{0} e^u\, du$$

$$= \frac{1}{2} e^u \Big|_{-9}^{0}$$

$$= \frac{1}{2}\left(1 - e^{-9}\right)$$

part	sign
a.	+
b.	+
c.	−
d.	+
e.	0
f.	0
g.	−

Problem 1. Let D be the region inside the unit circle in $\mathbb{R}^2$, let R be the right half of D, and let B be the bottom half of D. Determine whether each of the following double integrals is positive, negative, or zero by thinking about the (signed) volume that each represents and whether the function in the integrand is positive, negative, or zero over the region of integration. You might find it helpful to use CalcPlot3d.

a. $\iint_D 1\, dA$

b. $\iint_D (1 - \sqrt{x^2+y^2})\, dA$

c. $\iint_D (-1 + x^2 + y^2)\, dA$

d. $\iint_R x\, dA$

e. $\iint_R y\, dA$

f. $\iint_B x\, dA$

g. $\iint_B y\, dA$

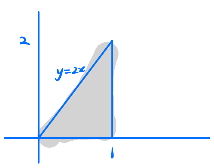
Problem 2. Each double integral below cannot be computed using the given order of integration. Sketch the region of integration, set up the integral again with the reversed order of integration, and compute its value.

a. $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx dy$

b. $\int_0^1 \int_{\sqrt{y}}^1 e^{-x^3}\, dx dy$

c. $\int_{-1}^0 \int_{-y}^1 \sin(x^2)\, dx dy + \int_0^1 \int_y^1 \sin(x^2)\, dx dy$

(a)



$$\int_{x=0}^{x=1} \int_{y=0}^{y=2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 e^{x^2} \left(y \Big|_0^{2x} \right) dx$$

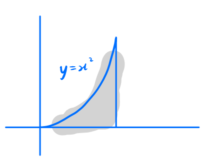
$$= \int_0^1 2x e^{x^2}\, dx \qquad u = x^2$$

$$du = 2x\,dx$$

$$= \int_0^1 e^u\, du$$

$$= e^u \Big|_0^1 = e - 1$$

(b)



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} e^{-x^3}\, dy\, dx$$

$$= \int_0^1 e^{-x^3} \left(y \Big|_0^{x^2} \right) dx$$

$$= \int_0^1 x^2 e^{-x^3}\, dx \qquad u = -x^3$$

$$du = -3x^2 dx$$

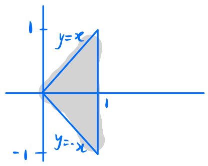
$$-\tfrac{1}{3} du = x^2 dx$$

$$= -\frac{1}{3} \int_0^{-1} e^u\, du$$

$$= \frac{-1}{3} e^u \Big|_0^{-1}$$

$$= \frac{1}{3}(1 - e^{-1})$$

(c)



$$\int_{x=0}^{x=1} \int_{y=-x}^{y=x} \sin(x^2)\,dy\,dx = \int_0^1 \sin(x^2)\left(y\Big|_{-x}^{x} \right) dx$$

$$= \int_0^1 2x\sin(x^2)\,dx \qquad u = x^2$$

$$du = 2x\,dx$$

$$= \int_0^1 \sin u\,du$$

$$= -\cos u \Big|_0^1$$

$$= 1 - \cos(1)$$

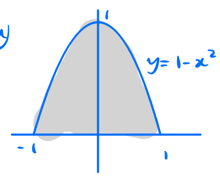
Problem 3. For each double integral below, sketch the region of integration and set up the integral with the reversed order of integration.

a. $\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,dxdy$

b. $\int_0^1 \int_{y^2}^{(y+1)/2} f(x,y)\,dxdy$

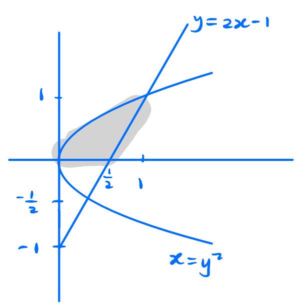
c. $\int_{-1}^1 \int_{-1+y^2}^{\sqrt{1-y^2}} f(x,y)\,dxdy$

(a)



$$\int_{x=-1}^{x=1} \int_{y=0}^{y=1-x^2} f(x,y)\,dydx$$

(b)



Intersection points:

$$\begin{cases} x = y^2 \\ x = \frac{y+1}{2} \end{cases} \Rightarrow y^2 = \frac{y+1}{2}$$

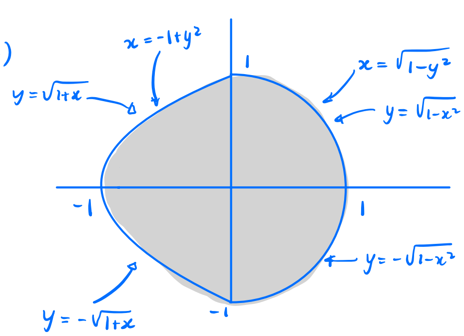
$$\Rightarrow 2y^2 - y - 1 = 0$$

$$\Rightarrow (2y+1)(y-1) = 0$$

$$\Rightarrow y = -\tfrac{1}{2},\ y = 1$$

$$\int_{x=0}^{x=\frac{1}{2}} \int_{y=0}^{y=\sqrt{x}} f(x,y)\,dydx + \int_{x=\frac{1}{2}}^{x=1} \int_{y=2x-1}^{y=\sqrt{x}} f(x,y)\,dydx$$

c)



$$\int_{x=-1}^{x=0} \int_{y=-\sqrt{1+x}}^{y=\sqrt{1+x}} 1\,dydx$$

$$+ \int_{x=0}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} 1\,dydx$$