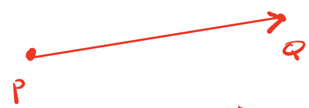
10.2 Introduction to Vectors

Informal idea A vector is a mathematical object that encodes both magnitude and direction (for example velocity).

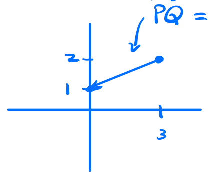
Definition A vector is a directed line segment between two points P and Q (these points might be in $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^n$).



vector $\overrightarrow{PQ}$ points from P to Q.

Example Let $P = (3,2)$ and $Q = (0,1)$. Plot $\overrightarrow{PQ}$, write it in component form, find its magnitude

$\overrightarrow{PQ}$ = arrow from P to Q



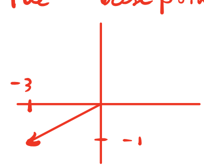
$$\overrightarrow{PQ} = \langle \underbrace{0-3}, \underbrace{1-2} \rangle = \langle -3,-1 \rangle$$

subtract x and y components

angled brackets for vectors

magnitude: $\|\overrightarrow{PQ}\| = \sqrt{(-3)^2 + (-1)^2}$

Remark Can also move the basepoint of $\overrightarrow{PQ}$ when plotting:



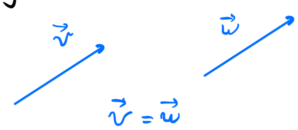
what matters is
- magnitude
- direction

Properties

① If $P=(a,b)$ and $Q=(c,d)$, $\overrightarrow{PQ} = \langle c-a, d-b\rangle$

② If $\vec{v} = \langle v_1, v_2\rangle$, then $\|\vec{v}\| = \sqrt{v_1^2+v_2^2}$

③ Two vectors are equal if they have the same magnitude and direction

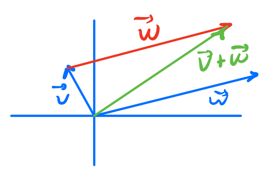


Some basic algebra of vectors

Let $\vec{v} = \langle v_1, v_2\rangle$, $\vec{w} = \langle w_1, w_2\rangle$.

Their <u>sum</u> is the vector $\vec{v}+\vec{w} = \langle v_1+w_1, v_2+w_2\rangle$.

geometric intuition:



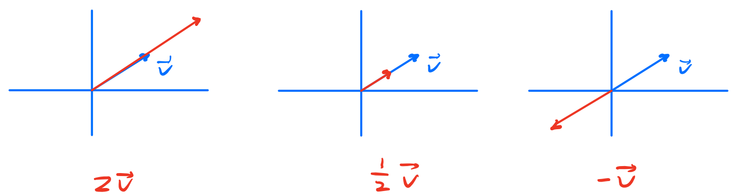
To plot $\vec{v}+\vec{w}$, slide $\vec{w}$ so its base is at the tip of $\vec{v}$. Then $\vec{v}+\vec{w}$ will connect the base of $\vec{v}$ to the tip of $\vec{w}$.

Let $c \in \mathbb{R}$. The scalar product of c and $\vec{v}$

the vector $c\vec{v} = \langle cv_1, cv_2 \rangle$

geometric intuition:

$c\vec{v}$ is a stretched or compressed version of $\vec{v}$



Important ideas

① $\vec{v}$ and $\vec{w}$ are parallel if there is a scalar c such that $\vec{v} = c\vec{w}$

② $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$.

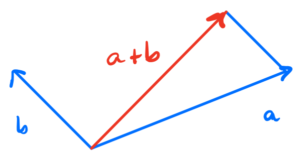
(since
$$\begin{aligned} \|c\vec{v}\| &= \sqrt{(cv_1)^2 + (cv_2)^2} \\ &= \sqrt{c^2(v_1^2 + v_2^2)} \\ &= \sqrt{c^2}\,\sqrt{v_1^2 + v_2^2} \\ &= |c|\,\|\vec{v}\|. \end{aligned}$$

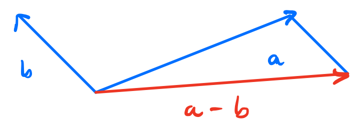
③ A unit vector is a vector of length 1.

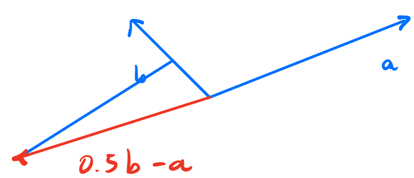
Given a vector $\vec{v}$, there are two unit vectors parallel to $\vec{v}$:

$$\frac{1}{\|\vec{v}\|}\vec{v} \quad \text{and} \quad -\frac{1}{\|\vec{v}\|}\vec{v}.$$

Problem 1. Consider the vectors $\mathbf{a}$ and $\mathbf{b}$ below. Draw the vectors $\mathbf{a}+\mathbf{b}, \mathbf{a}-\mathbf{b}, 0.5\mathbf{b}-\mathbf{a}$.





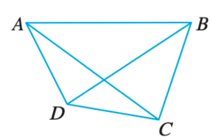


Problem 2. Write the following sums of vectors each as a single vector:

a. $\overrightarrow{AB}+\overrightarrow{BC}$

b. $\overrightarrow{CD}+\overrightarrow{DB}$

c. $\overrightarrow{DB}-\overrightarrow{AB}$



(a) $\overrightarrow{AC}$

(b) $\overrightarrow{CB}$

(c) $\overrightarrow{DA}$

Problem 3. Consider the vectors $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle -1, -4\rangle$.

a. Sketch $\mathbf{u}, \mathbf{v}$ on the same axes, using $(0,0)$ as the basepoint for both.

b. Add the vector $\mathbf{u} - \mathbf{v}$ to your sketch by scaling and shifting $\mathbf{v}$.

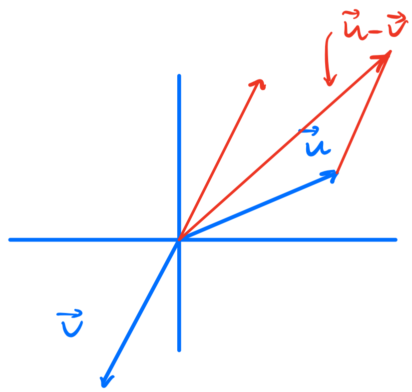
c. Compute $\mathbf{u} - \mathbf{v}$ in component form. Check that it matches your picture.

d. Find the unit vector in the direction of $\mathbf{u}$.

e. Find the unit vector in the direction opposite of $\mathbf{v}$.

f. Find the vector of length 3 in the direction oppositve of $\mathbf{u}$.

(a), (b)



(c) $\vec{u} - \vec{v} = \langle 2, 1\rangle - \langle -1, -4\rangle$

$= \langle 3, 5\rangle$

(d) $\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{5}}\langle 2, 1\rangle = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$

(e) $\frac{-1}{\|\vec{v}\|}\vec{v} = \frac{-1}{\sqrt{17}}\langle -1, -4\rangle = \langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\rangle$

(f) $\frac{-3}{\|\vec{u}\|}\vec{u} = \langle \frac{-6}{\sqrt{5}}, \frac{-3}{\sqrt{5}}\rangle$

Problem 4   Let $P = (4,3)$, $Q = (-1,5)$.

① Find $\vec{PQ}$ and $\vec{QP}$ in component form

② Suppose $\vec{v} = \vec{PR}$ and $\vec{v} = \langle 10,3 \rangle$. What is R?

③ Are $\vec{PR}$ and $\vec{PQ}$ parallel?

① $\vec{PQ} = \langle -1-4, 5-3 \rangle$

$= \langle -5, 2 \rangle$

$\vec{QP} = -\vec{PQ} = \langle 5, -2 \rangle$

② $R = (x,y)$ and

$$\langle 10, 3 \rangle = \vec{PR} = \langle x-4, y-3 \rangle$$

so $x = 14$, $y = 6$ and $R = (14, 6)$

③ If $\vec{PR}$ and $\vec{PQ}$ are parallel then there is a scalar c so that

$$\vec{PR} = c\vec{PQ}$$

$$\langle 10, 3 \rangle = c \langle -5, 2 \rangle$$

$$= \langle -5c, 2c \rangle$$

But this is not possible since $-5c = 10$ means $c = -2$ but $3 = 2c = 2(-2) = -4$ is not valid.