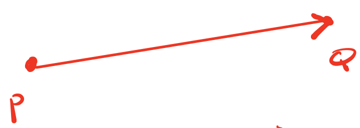
10.2 Introduction to Vectors

<u>Informal idea</u> A vector is a mathematical object that encodes both magnitude and direction (for example relative position, velocity, force)

<u>Definition</u> A <u>vector</u> is a <u>directed</u> line segment between two points P and Q (these points might be in $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^n$).

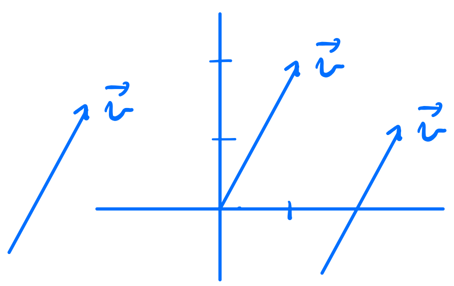


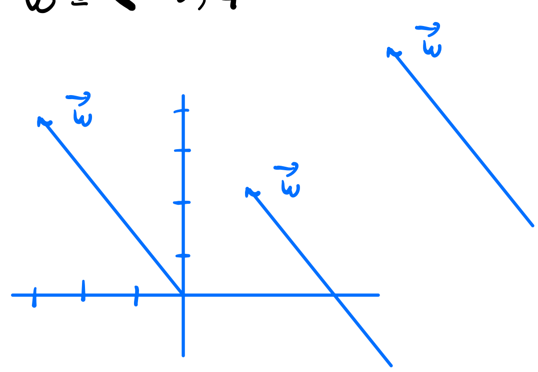
vector $\overrightarrow{PQ}$ points from P to Q.

We typically express vectors in component form:

angle braces denote vectors! → $\langle a, b \rangle$ or $\langle a, b, c \rangle$

<u>Example</u> Plot $\vec{v} = \langle 1, 2 \rangle$, $\vec{w} = \langle -3, 4 \rangle$

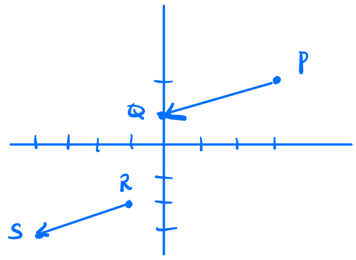




Since vectors only encode magnitude and direction, we can plot them with base points other than origin.

Example Let $P=(3,2)$, $Q=(0,1)$, $R=(-1,-2)$, $S=(-4,-3)$.

Plot $\overrightarrow{PQ}$ and $\overrightarrow{RS}$, write them in component form and find their magnitudes (ie lengths).



$\overrightarrow{PQ} = \langle 0-3, 1-2 \rangle = \langle -3, -1 \rangle$

$\overrightarrow{RS} = \langle -4-(-1), -3-(-2) \rangle = \langle -3, -1 \rangle$

They're the same vector!

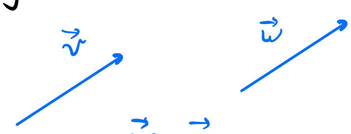
Magnitude: $\| \overrightarrow{PQ} \| = \sqrt{(-3)^2 + (-1)^2}$

Properties

① If $P=(a,b)$ and $Q=(c,d)$, $\overrightarrow{PQ} = \langle c-a, d-b \rangle$

② If $\vec{v} = \langle v_1, v_2 \rangle$, then $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$

③ Two vectors are equal if they have the same magnitude and direction



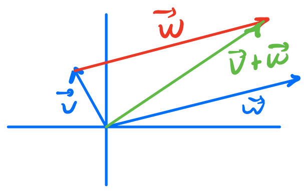
$\vec{v} = \vec{w}$

Some basic algebra of vectors

Let $\vec{v} = \langle v_1, v_2 \rangle$, $\vec{w} = \langle w_1, w_2 \rangle$.

Their sum is the vector $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$.

geometric intuition:



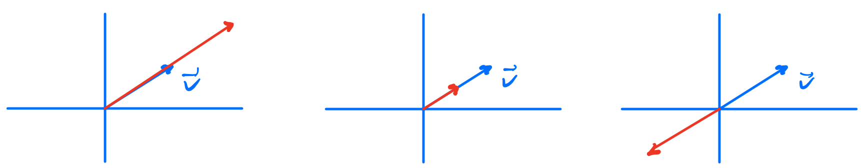
To plot $\vec{v}+\vec{w}$, slide $\vec{w}$ so its base is at the tip of $\vec{v}$. Then $\vec{v}+\vec{w}$ will connect the base of $\vec{v}$ to the tip of $\vec{w}$.

Remark $\vec{v}+\vec{w}$ is like describing a walking route: first walk along $\vec{v}$, from there walk along $\vec{w}$.

Let $c \in \mathbb{R}$. The scalar product of c and $\vec{v}$ the vector $c\vec{v} = \langle cv_1, cv_2 \rangle$

geometric intuition:

$c\vec{v}$ is a stretched or compressed version of $\vec{v}$



$2\vec{v}$ $\qquad$ $\frac{1}{2}\vec{v}$ $\qquad$ $-\vec{v}$

Important ideas

① $\vec{v}$ and $\vec{w}$ are parallel if there is a scalar c such that $\vec{v} = c\vec{w}$

② $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$.

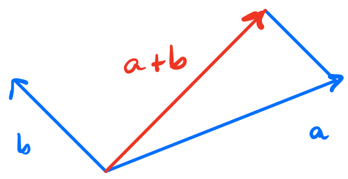
(since
$$\|c\vec{v}\| = \sqrt{(cv_1)^2 + (cv_2)^2}$$
$$= \sqrt{c^2(v_1^2 + v_2^2)}$$
$$= \sqrt{c^2}\sqrt{v_1^2 + v_2^2}$$
$$= |c|\,\|\vec{v}\|.$$

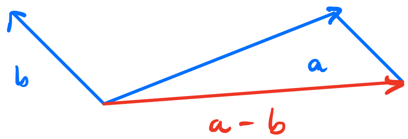
③ A unit vector is a vector of length 1.

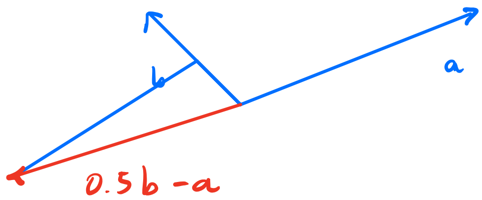
Given a vector $\vec{v}$, there are two unit vectors parallel to $\vec{v}$:

$$\frac{1}{\|\vec{v}\|}\vec{v} \quad \text{and} \quad -\frac{1}{\|\vec{v}\|}\vec{v}.$$

Problem 1. Consider the vectors **a** and **b** below. Draw the vectors $\mathbf{a}+\mathbf{b}, \mathbf{a}-\mathbf{b}, 0.5\mathbf{b}-\mathbf{a}$.





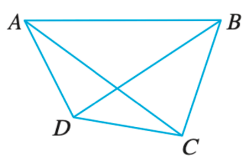


Problem 2. Write the following sums of vectors each as a single vector:

a. $\overrightarrow{AB}+\overrightarrow{BC}$

b. $\overrightarrow{CD}+\overrightarrow{DB}$

c. $\overrightarrow{DB}-\overrightarrow{AB}$



(a) $\overrightarrow{AC}$

(b) $\overrightarrow{CB}$

(c) $\overrightarrow{DA}$

Problem 3. Consider the vectors $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle -1, -4\rangle$.

a. Sketch $\mathbf{u}, \mathbf{v}$ on the same axes, using $(0, 0)$ as the basepoint for both.

b. Add the vector $\mathbf{u} - \mathbf{v}$ to your sketch by scaling and shifting $\mathbf{v}$.

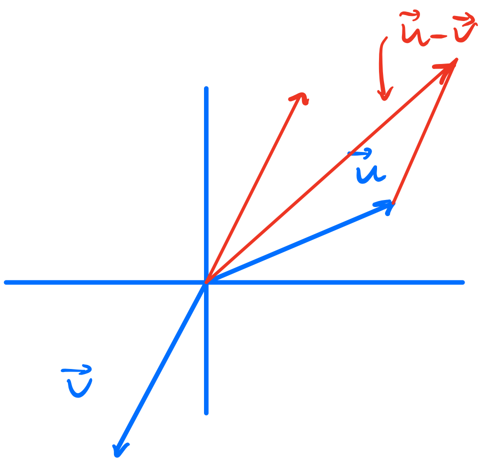
c. Compute $\mathbf{u} - \mathbf{v}$ in component form. Check that it matches your picture.

d. Find the unit vector in the direction of $\mathbf{u}$.

e. Find the unit vector in the direction opposite of $\mathbf{v}$.

f. Find the vector of length 3 in the direction oppositve of $\mathbf{u}$.

(a), (b)



(c) $\vec{u} - \vec{v} = \langle 2, 1\rangle - \langle -1, -4\rangle$

$= \langle 3, 5\rangle$

(d) $\dfrac{1}{\|\vec{u}\|}\vec{u} = \dfrac{1}{\sqrt{5}}\langle 2, 1\rangle = \left\langle \dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right\rangle$

(e) $\dfrac{-1}{\|\vec{v}\|}\vec{v} = \dfrac{-1}{\sqrt{17}}\langle -1, -4\rangle = \left\langle \dfrac{1}{\sqrt{17}}, \dfrac{4}{\sqrt{17}}\right\rangle$

(f) $\dfrac{-3}{\|\vec{u}\|}\vec{u} = \left\langle \dfrac{-6}{\sqrt{5}}, \dfrac{-3}{\sqrt{5}}\right\rangle$

Problem 4 Let $P = (4, 3)$, $Q = (-1, 5)$.

① Find $\vec{PQ}$ and $\vec{QP}$ in component form

② Suppose $\vec{v} = \vec{PR}$ and $\vec{v} = \langle 10, 3 \rangle$. What is R?

③ Are $\vec{PR}$ and $\vec{PQ}$ parallel?

① $\vec{PQ} = \langle -1-4, 5-3 \rangle$

$= \langle -5, 2 \rangle$

$\vec{QP} = -\vec{PQ} = \langle 5, -2 \rangle$

② $R = (x, y)$ and

$\langle 10, 3 \rangle = \vec{PR} = \langle x-4, y-3 \rangle$

so $x = 14$, $y = 6$ and $R = (14, 6)$

③ If $\vec{PR}$ and $\vec{PQ}$ are parallel then there is a scalar c so that

$$\vec{PR} = c\vec{PQ}$$

$$\langle 10, 3 \rangle = c \langle -5, 2 \rangle$$

$$= \langle -5c, 2c \rangle$$

But this is not possible since $-5c = 10$ means $c = -2$ but $3 = 2c = 2(-2) = -4$ is not valid.