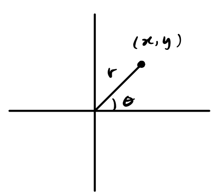
9.4 Intro to Polar Coordinates

Given (x, y) in $\mathbb{R}^2$ it's possible to express it in a new coordinate system (r, θ) called polar coordinates, where

r represents the distance from (x, y) to $(0,0)$

and θ represents the angle between $\langle x, y \rangle$ and the positive x-axis.



$$\boxed{\begin{gathered} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ x = r\cos\theta \\ y = r\sin\theta \end{gathered}}$$

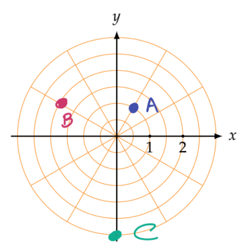
Rule of Thumb

① We'll use the convention $r \geq 0$ but when $r < 0$ we have $(r, \theta) = (-r, \theta + \pi)$

② We often think of θ being in the range $[0, 2\pi)$, though sometimes we use $[-\pi, \pi)$

Example Given points in polar coordinates, express them in Cartesian coordinates.

polar	Cartesian
$A = (1, \frac{\pi}{3})$	$(\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3})) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
$B = (2, \frac{5\pi}{6})$	$(2\cos(\frac{5\pi}{6}), 2\sin(\frac{5\pi}{6})) = (\frac{-2\sqrt{3}}{2}, 2(\frac{1}{2})) = (-\sqrt{3}, 1)$
$C = (3, \frac{3\pi}{2})$	$(3\cos(\frac{3\pi}{2}), 3\sin(\frac{3\pi}{2})) = (0, -3)$



Example Given points in Cartesian coordinates, express them in polar coordinates.

Cartesian	Polar
$P = (1, \sqrt{3})$	$(2, \pi/3)$
$Q = (-3, 0)$	$(3, \pi)$
$R = (-2, -2)$	$(\sqrt{8}, \frac{5\pi}{4}) = (\sqrt{8}, \frac{-3\pi}{4})$

P $r = \sqrt{1^2 + \sqrt{3}^2} = 2$

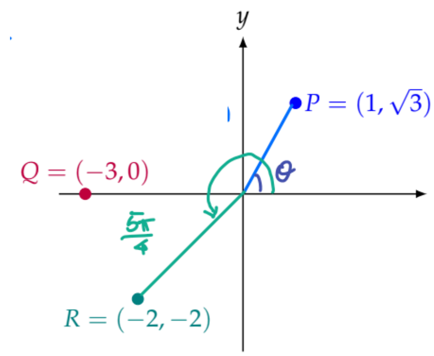
$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

Q $r = \sqrt{(-3)^2 + 0^2}$

$\tan\theta = \frac{0}{3} = 0$

R $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$

$\tan\theta = \frac{-2}{-2} = 1$



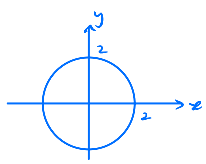
Warning Your calculator's $\tan^{-1}$ function only outputs angles in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so you have to be careful when finding polar coordinates of points in quadrants II and III.

Example What graph do the following polar equations represent in the xy-plane?

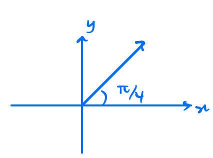
ⓐ $r = 2$

ⓑ $\theta = \pi/4$, $r \geq 0$

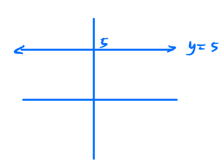
ⓒ $r \sin\theta = 5$



$r = 2$



$\theta = \pi/4$,
$r \geq 0$



$r \sin\theta = 5$

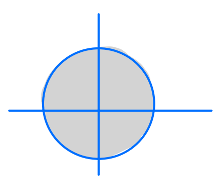
<u>Example</u> Sketch the regions in the xy-plane represented by the following polar expressions.

ⓐ $0 \le r \le 1$

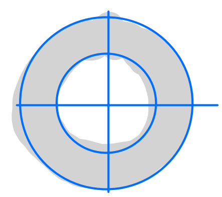
ⓑ $1 \le r \le 2$

ⓒ $0 \le \theta \le \pi/4$

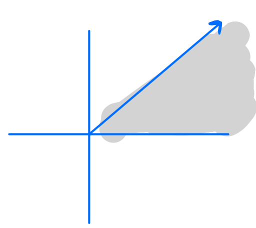
ⓓ $-\pi/4 \le \theta \le \pi/4$, $1 \le r \le 2$



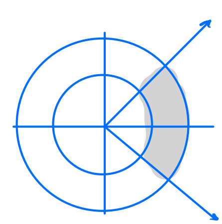
$r \le 1$



$1 \le r \le 2$



$0 \le \theta \le \frac{\pi}{4}$



$1 \le r \le 2$

$-\pi/4 \le \theta \le \pi/4$

Problem 1. For each point given in Cartesian coordinates, find a polar coordinate representation. Likewise, for each point in polar coodinates, give its Cartesian coordinates.

Cartesian	Polar	Polar	Cartesian
$(1, -1)$	$(\sqrt{2}, -\frac{\pi}{4})$	$(5, \pi)$	$(-5, 0)$
$(-4, 0)$	$(4, \pi)$	$(2, 7\pi/4)$	$(\sqrt{2}, -\sqrt{2})$
$(-\sqrt{2}/2, \sqrt{2}/2)$	$(1, 3\pi/4)$	$(1, -5\pi/4)$	$(-\sqrt{2}/2, \sqrt{2}/2)$
$(\sqrt{3}/2, 1/2)$	$(1, \pi/6)$	$(3, -5\pi/6)$	$(-3\sqrt{3}/2, -3/2)$

Problem 2. Sketch the regions described by the following polar inequalities or equations.

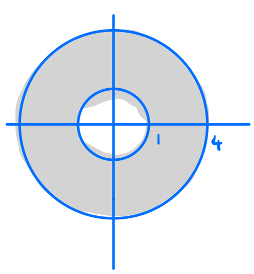
a. $1 \le r \le 4$

b. $\pi/4 \le \theta \le 3\pi/4$, $r \ge 0$

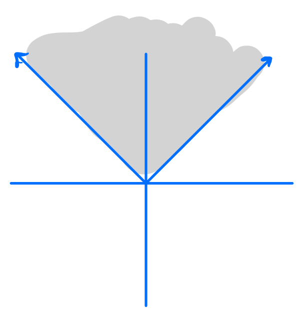
c. $0 \le r \le 1, \pi/2 \le \theta \le 3\pi/2$

d. $1 \le r \le 2, 0 \le \theta \le 2\pi/3$.

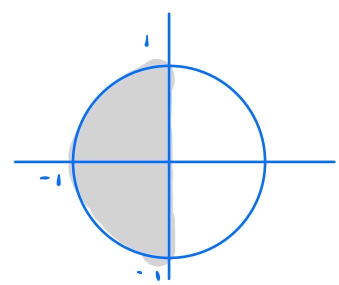
e. $4 \le r \le 9, -3\pi/4 \le \theta \le 3\pi/4$



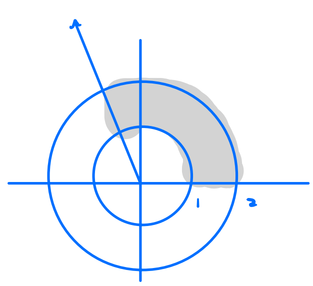
(a)



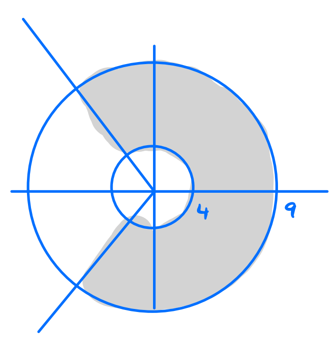
(b)



(c)



(d)



(e)

Problem 3. Sketch the following regions and express them using inequalities involving r and θ. Assume all circles are centered at the origin.

a. The region in the first quadrant enclosed by a quarter circle of radius 3.

b. The region in the third and fourth quadrants enclosed by a half circle of radius 2.

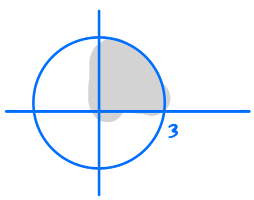
c. The region in the first and fourth quadrants enclosed by a half circle of radius 1.

d. The annulus inside a circle of radius 5 and outside a circle of radius 2.

e. The quarter annulus in the second quadrant inside a circle of radius 2 and outside a circle of radius 1.

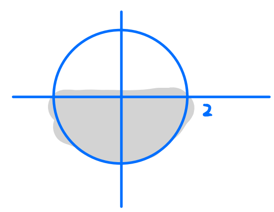
f. The quarter annulus in the top half of the xy-plane between the lines $y = \pm x$ and inside the circle of radius 2 and outside the circle of radius 1.

(a) $0 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}$

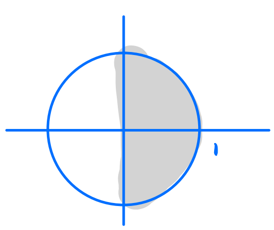


(b) $0 \le r \le 2, \quad -\pi \le \theta \le 0$

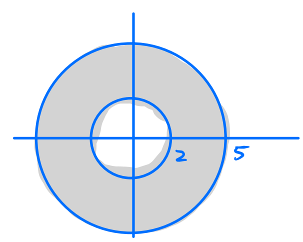
(or $\pi \le \theta \le 2\pi$)



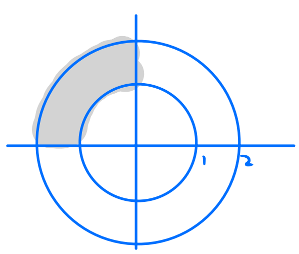
(c) $0 \le r \le 1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



(d) $2 \le r \le 5$



(e) $1 \le r \le 2, \quad \frac{\pi}{2} \le \theta \le \pi$



(f) $1 \le r \le 2, \quad \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$