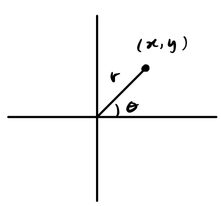
9.4 Intro to Polar Coordinates

Given (x,y) in $\mathbb{R}^2$ it's possible to express it in a new coordinate system (r,θ) called polar coordinates, where r represents the distance from (x,y) to $(0,0)$ and θ represents the angle between $\langle x,y \rangle$ and the positive x-axis.



$$\boxed{\begin{aligned} r^2 &= x^2+y^2 \\ \tan\theta &= \frac{y}{x} \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

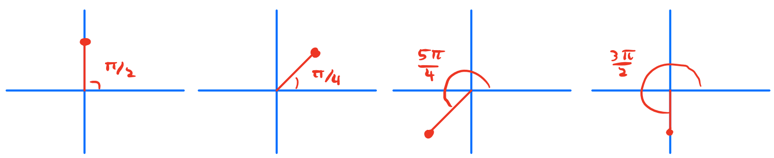
Rule of Thumb

① We'll use the convention $r \geq 0$

② We often think of θ being in the range $[0, 2\pi)$, though sometimes we use $[-\pi, \pi)$

Example Given points in Cartesian coordinates, express them in polar coordinates.

Cartesian	Polar
$(0,1)$	$(1, \frac{\pi}{2})$
$(1,1)$	$(\sqrt{2}, \frac{\pi}{4})$
$(-1,-1)$	$(\sqrt{2}, \frac{5\pi}{4})$ or $(\sqrt{2}, \frac{-3\pi}{4})$
$(0,-1)$	$(1, \frac{3\pi}{2})$ or $(1, \frac{-\pi}{2})$



Example Given points in polar coordinates, express them in Cartesian coordinates.

Polar	Cartesian
$(1, \frac{2\pi}{3})$	$\left(\cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right)\right) = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$(3, \frac{\pi}{4})$	$\left(3\cos\left(\frac{\pi}{4}\right), 3\sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$
$(1, \frac{-\pi}{6})$	$\left(\cos\left(\frac{-\pi}{6}\right), \sin\left(\frac{-\pi}{6}\right)\right) = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

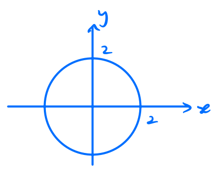
Example What graph do the following polar equations represent in the xy-plane?

(a) $r = 2$

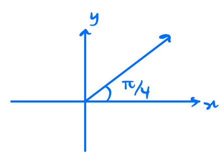
(b) $\theta = \pi/4$

(c) $r\sin\theta = 5$

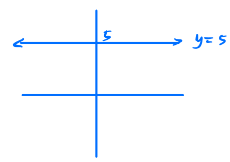
(d) $2\cos\theta = r$



$r = 2$



$\theta = \pi/4$



$r\sin\theta = 5$

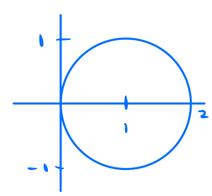
$$2\cos\theta = r \implies 2r\cos\theta = r^2$$

$$\implies 2x = x^2 + y^2$$

$$\implies 0 = x^2 - 2x + y^2$$

$$1 = x^2 - 2x + 1 + y^2$$

$$= (x-1)^2 + y^2$$



Example: Sketch the regions in the xy-plane represented by the following polar expressions.

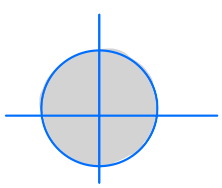
ⓐ $r \le 1$

ⓑ $1 \le r \le 2$

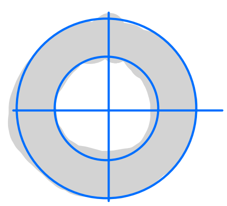
ⓒ $0 \le \theta \le \pi/4$

ⓓ $-\pi/4 \le \theta \le \pi/4$, $1 \le r \le 2$

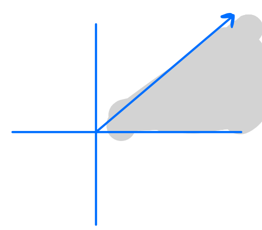
ⓔ $-\pi/4 \le \theta \le \pi/4$, $r \le 2\cos\theta$



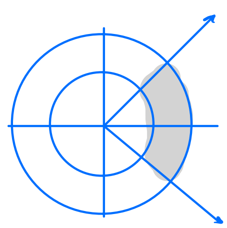
$r \le 1$



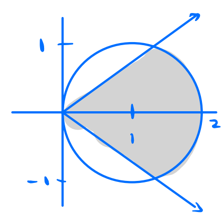
$1 \le r \le 2$



$0 \le \theta \le \frac{\pi}{4}$



$1 \le r \le 2$



Problem 1. For each point given in Cartesian coordinates, find a polar coordinate representation. Likewise, for each point in polar coodinates, give its Cartesian coordinates.

Cartesian	Polar	Polar	Cartesian
$(1,-1)$	$(\sqrt{2}, -\frac{\pi}{4})$	$(5,\pi)$	$(-5,0)$
$(-4,0)$	$(4, \pi)$	$(2,5\pi/4)$	$(-\sqrt{2}, -\sqrt{2})$
$(-\sqrt{2}/2, \sqrt{2}/2)$	$(1, 3\pi/4)$	$(1,-3\pi/4)$	$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$
$(\sqrt{3}/2, 1/2)$	$(1, \pi/6)$	$(3,5\pi/6)$	$(-\frac{3\sqrt{3}}{2}, \frac{3}{2})$
$(-\sqrt{3},1)$	$(2, 5\pi/6)$	$(3,-5\pi/6)$	$(-\frac{3\sqrt{3}}{2}, -\frac{3}{2})$

Problem 2. Sketch the regions described by the following polar inequalities or equations.

a. $1 \leq r \leq 4$

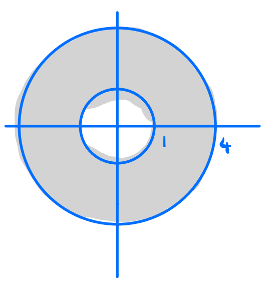
b. $\pi/4 \leq \theta \leq 3\pi/4$

c. $r \leq 1, \pi/2 \leq \theta \leq 3\pi/2$

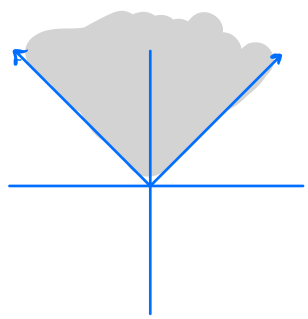
d. $1 \leq r \leq 2, 0 \leq \theta \leq 2\pi/3$.

e. $4 \leq r \leq 9, -3\pi/4 \leq \theta \leq 3\pi/4$

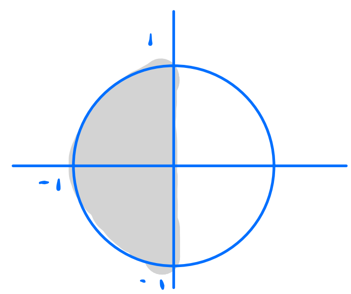
f. $r \leq 2\sin\theta, \pi/4 \leq \theta \leq 3\pi/4$



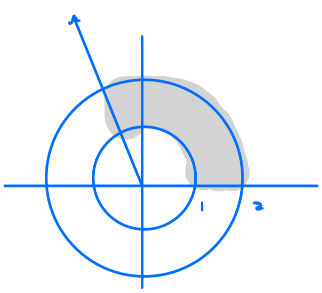
(a)



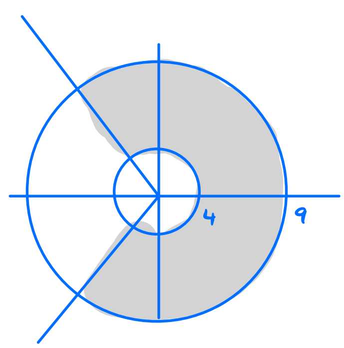
(b)



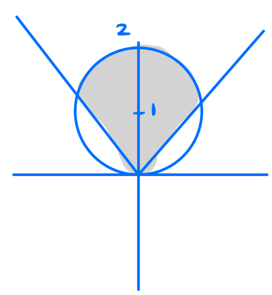
(c)



(d)



(e)



(f)

Problem 3. Sketch the following regions and express them using inequalities involving r and θ. Assume all circles are centered at the origin.

a. The region in the first quadrant enclosed by a quarter circle of radius 3.

b. The region in the third and fourth quadrants enclosed by a half circle of radius 2.

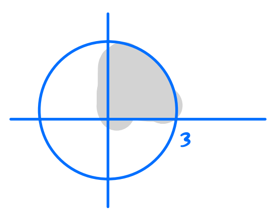
c. The region in the first and fourth quadrants enclosed by a half circle of radius 1.

d. The annulus inside a circle of radius 5 and outside a circle of radius 2.

e. The quarter annulus in the second quadrant inside a circle of radius 2 and outside a circle of radius 1.

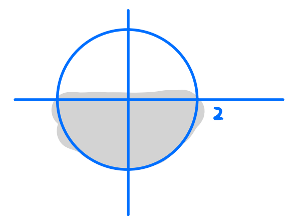
f. The quarter annulus in the top half of the xy-plane between the lines $y = \pm x$ and inside the circle of radius 2 and outside the circle of radius 1.

(a) $r \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}$

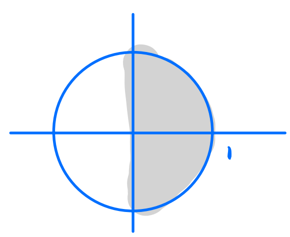


(b) $r \leq 2, \quad -\pi \leq \theta \leq 0$

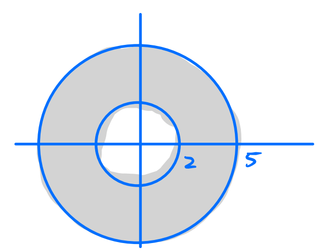
$(\text{or} \quad \pi \leq \theta \leq 2\pi)$



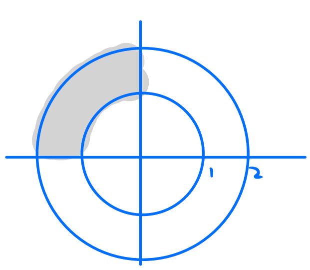
(c) $r \leq 1, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



(d) $2 \leq r \leq 5$



(e) $1 \leq r \leq 2, \quad \frac{\pi}{2} \leq \theta \leq \pi$



(f) $1 \leq r \leq 2, \quad \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$