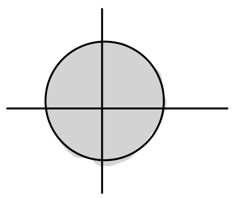
13.3 Polar Integrals

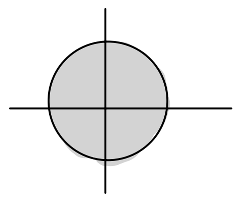
Goal Compute $\iint_R f(x,y)\,dA$ when R is a region in the xy-plane that is easier to describe in polar coordinates than Cartesian coordinates.



unit disk in Cartesian:

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$



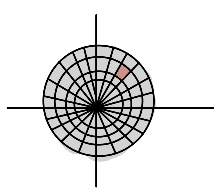
unit disk in polar:

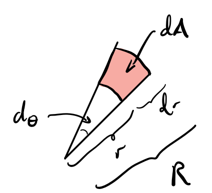
$0 \le \theta \le 2\pi$

$0 \le r \le 1$

constant bounds are nicer to work with in integrals

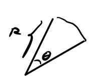
$\boxed{\text{Warning In Cartesian coordinates } dA \text{ is } dy\,dx \text{ or } dx\,dy. \text{ In polar coordinates } dA \text{ is } r\,dr\,d\theta.}$





area of disk with radius R $= \pi R^2$

area of wedge of disk with angle θ $= \dfrac{\theta}{2\pi}(\pi R^2) = \dfrac{\theta}{2} R^2$



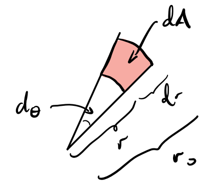
$$\text{area } dA = \frac{1}{2} R^2 \, d\theta - \frac{1}{2} r^2 \, d\theta$$

$$= \frac{d\theta}{2} (R^2 - r^2)$$

$$= \frac{d\theta}{2} (R - r)(R + r)$$

$$= \frac{1}{2} \, dr \, d\theta \, (R + r)$$

$$\approx r \, dr \, d\theta \qquad \text{since } R \approx r$$



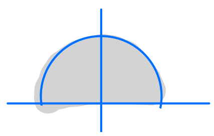
Conclusion

$$\iint_R f(x,y) \, dx \, dy = \iint_R f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$$

in Cartesian coordinates

in polar coordinates

Example Set up $\iint_R e^{-(x^2+y^2)}\,dA$ when R is the upper half of the unit disk. Use polar coordinates and then compare with Cartesian coordinates.



$0 \le r \le 1$

$0 \le \theta \le \pi$

$$\int_{x=-1}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx$$ (impossible to do by hand)

$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} e^{-r^2} \cdot r\,dr\,d\theta$$

$u = -r^2$

$du = -2r\,dr$

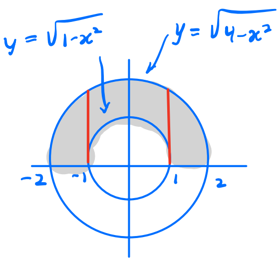
$-\frac{1}{2}\,du = r\,dr$

$$= -\frac{1}{2} \int_0^{\pi} \int_0^{-1} e^u\,du\,d\theta$$

$$= \frac{-1}{2} \int_0^{\pi} \left(e^u \Big|_0^{-1} \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi} (1-e^{-1})\,d\theta = \frac{\pi}{2}(1-e^{-1})$$

Example Set up $\displaystyle\iint_R \frac{1}{(x^2+y^2)^{3/2}}\,dA$ when R is the upper half of the annulus with inner and outer radiuses of 1 and 2. Use polar coordinates and compare with Cartesian.



$1 \le r \le 2$

$0 \le \theta \le \pi$

$$\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} \frac{1}{(r^2)^{3/2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 r^{-2}\,dr\,d\theta = \int_0^{\pi} \left(-r^{-1}\Big|_1^2\right) d\theta$$

$$= \int_0^{\pi} \left(1-\tfrac{1}{2}\right) d\theta = \frac{\pi}{2}$$

$$\int_{-2}^{-1}\int_0^{\sqrt{4-x^2}} \frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx + \int_1^2\int_0^{\sqrt{4-x^2}} \frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx$$

<u>Example</u> Compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Let $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Notice $I = \displaystyle\int_{-\infty}^{\infty} e^{-y^2}\,dy$ too.

Therefore

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \qquad u = -r^2$$

$$du = -2r\,dr$$

$$= \int_0^{2\pi}\int_0^{-\infty} -\tfrac{1}{2}e^{u}\,du\,d\theta \qquad -\tfrac{1}{2}du = r\,dr$$

$$= \int_0^{2\pi}\left(-\tfrac{1}{2}e^{u}\Big|_0^{-\infty}\right)d\theta$$

$$= \int_0^{2\pi}\tfrac{1}{2}\,d\theta$$

$$= \pi$$

So $I = \sqrt{\pi}$.

This is called a Gaussian integral. It's related to the normal distribution in statistics!

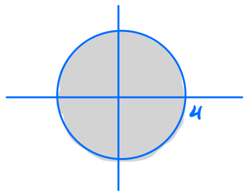
Problem 1. Set up both polar and Cartesian integrals for $\iint f(x,y)\,dA$ where f and R are given as follows.

a. $f(x,y) = 3 + 2x - 4y$, R is the disk of radius 4.

b. $f(x,y) = (x^2+y^2)^{-5/2}$, R is the left half the annulus with inner and outer radiuses 2 and 5.

c. $f(x,y) = e^{-(x^2+y^2)}$, R is the portion of the disk of radius 6 in the second and fourth quadrants.

(a)

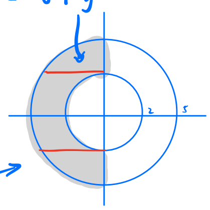


$$\int_0^{2\pi}\int_0^4 (3 + 2r\cos\theta - 4r\sin\theta)\, r\,dr\,d\theta$$

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (3 + 2x - 4y)\,dy\,dx$$

$x = -\sqrt{4-y^2}$

(b)



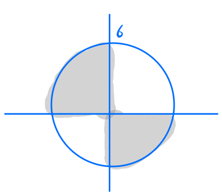
$x = -\sqrt{25-y^2}$

$$\int_{\pi/2}^{3\pi/2}\int_2^5 (r^2)^{-5/2}\, r\,dr\,d\theta$$

$$\int_{y=-5}^{y=-2}\int_{x=-\sqrt{25-y^2}}^{x=0} (x^2+y^2)^{-5/2}\,dx\,dy + \int_{y=-2}^{y=2}\int_{x=-\sqrt{25-y^2}}^{x=-\sqrt{4-y^2}} (x^2+y^2)^{-5/2}\,dx\,dy$$

$$+ \int_{y=2}^{y=5}\int_{x=-\sqrt{25-y^2}}^{x=0} (x^2+y^2)^{-5/2}\,dx\,dy$$

(c)



$$\int_{\pi/2}^{\pi}\int_0^6 e^{-r^2}\, r\,dr\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi}\int_0^6 e^{-r^2}\, r\,dr\,d\theta$$

$$\int_{-6}^{0}\int_0^{\sqrt{36-x^2}} e^{-(x^2+y^2)}\,dy\,dx + \int_0^6\int_{-\sqrt{36-x^2}}^{0} e^{-(x^2+y^2)}\,dy\,dx$$

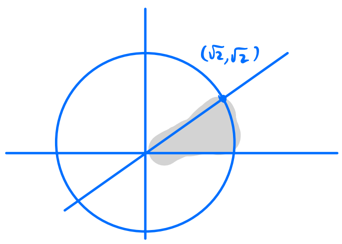
Problem 2. Rewrite the following integrals in polar coordinates:

a. $\int_0^{\sqrt{2}/2}\int_y^{\sqrt{1-y^2}} (x+y)\, dxdy$

b. $\int_{-\sqrt{2}/2}^{0}\int_{-x}^{\sqrt{1-x^2}} (x^2+y^2)\, dydx$

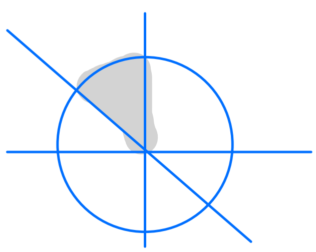
c. $\int_{-1}^{-\sqrt{2}/2}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3x-y)\, dydx + \int_{-\sqrt{2}/2}^{0}\int_{x}^{\sqrt{1-x^2}} (3x-y)\, dydx$

a



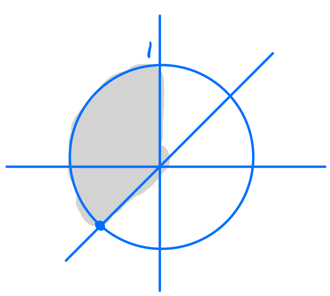
$$\int_0^{\pi/4}\int_0^1 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

b



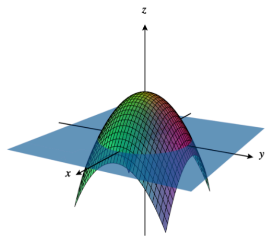
$$\int_{\pi/2}^{3\pi/4}\int_0^1 r^3\, dr\, d\theta$$

c



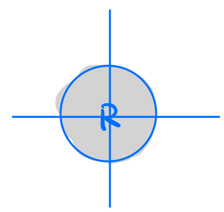
$$\int_{\pi/2}^{5\pi/4}\int_0^1 (3r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

Problem 3. Find the volume of the solid, shown below, bounded by paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.



The intersection of the paraboloid with $z=0$ (the xy-plane) is $x^2+y^2=1$. So the volume is

$$V = \iint_R (1-x^2-y^2)\,dA$$



where R is the unit disk. So

$$V = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4 \Big|_0^1\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\,d\theta$$

$$= \frac{\pi}{2}.$$