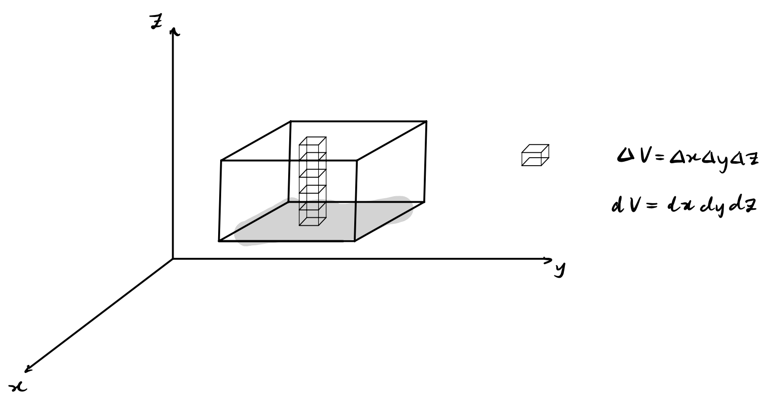
13.6 Triple Integrals

<u>Goal</u> Given a function $f(x,y,z)$ of three variables and a solid region $D \subseteq \mathbb{R}^3$, understand how to set up and compute $\iiint_D f(x,y,z)\, dV$.

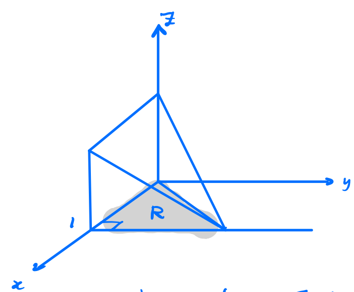
<u>Example</u> Suppose D is the rectangular solid where $a \le x \le b$, $c \le y \le d$, $p \le z \le q$ and $f(x,y,z)$ represents density (mass per unit volume). Set up a triple integral for the volume of D.



<u>Idea</u> subdivide D into small rectangular solids of volume $\Delta V = \Delta x \Delta y \Delta z$ whose mass at point (x,y,z) is $f(x,y,z)\Delta V$ and sum the results. As $\Delta x, \Delta y, \Delta z \to 0$, this becomes a <u>triple</u> <u>integral</u>.

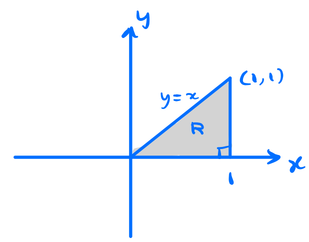
$$\iiint_D f(x,y,z)\, dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\, dz\, dy\, dx$$

Example Let D be the solid region that is bounded below by the triangular region in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$, and bounded above by the plane $z = 1-y$. Set up a triple integral for its mass, given density $f(x,y,z)$.



top surface: $z = 1-y$

bottom: $z = 0$



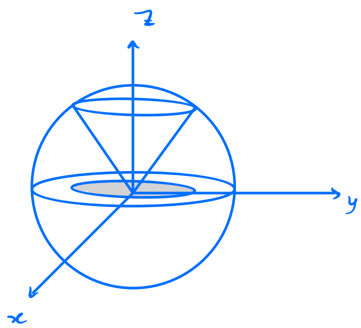
$$\underbrace{\int_{x=0}^{x=1} \int_{y=0}^{y=x}}_{\substack{\text{double integral} \\ \text{over } R}} \underbrace{\int_{z=0}^{z=1-y}}_{\substack{\text{top and} \\ \text{bottom surfaces}}} f(x,y,z)\, dz\, dy\, dx$$

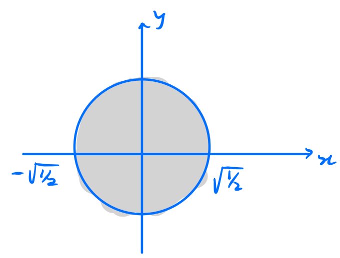
Guidelines for triple integrals

1. outer limits should be constants
2. middle limits can only depend on outer variable
3. inner limits can depend on outer and middle variables

Example Let D be the solid region that is bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the hemisphere $z = \sqrt{1-x^2-y^2}$.

Set up a triple integral for its volume.





Intersection:

$$\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2}$$

$$\Rightarrow \quad x^2+y^2 = 1-x^2-y^2$$

$$\Rightarrow \quad x^2+y^2 = \tfrac{1}{2} \quad (\text{circle of radius } \sqrt{1/2})$$

$$\int_{x=-\sqrt{1/2}}^{x=\sqrt{1/2}} \int_{y=-\sqrt{1/2-x^2}}^{y=\sqrt{1/2-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{1-x^2-y^2}} 1 \, dz\, dy\, dx$$

Guide to triple integrals with z inside limits

1. Sketch D
2. Collapse D onto region R in xy-plane and sketch R
3. Outer and middle limits are double integral over R, inner limits are top and bottom surfaces of D

Problem 1. Let D be the solid rectangular region given by $0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3$. Suppose the region has density $f(x, y, z) = 1 + xyz$. Find the mass of the region.

$$\int_0^2 \int_0^1 \int_0^3 (1 + xyz)\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^1 \left(3 + \tfrac{1}{2} xyz^2 \Big|_0^3\right) dy\, dx$$

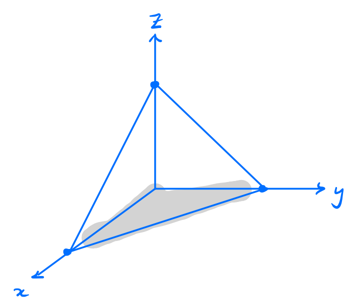
$$= \int_0^2 \int_0^1 \left(3 + \tfrac{9}{2} xy\right) dy\, dx$$

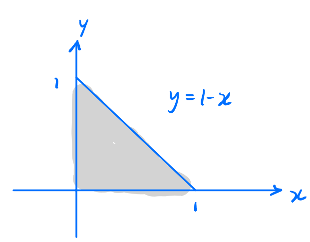
$$= \int_0^2 \left(3 + \tfrac{9}{4} x\right) dx$$

$$= 6 + \tfrac{9}{8} x^2 \Big|_0^2$$

$$= 6 + \tfrac{9}{2} = 10.5$$

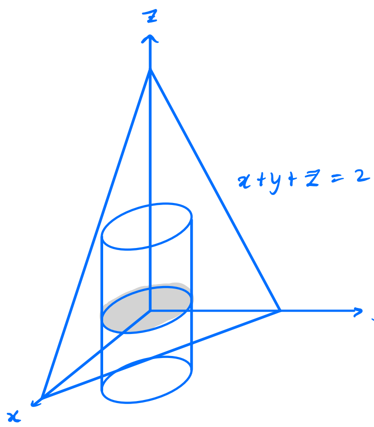
Problem 2. Let D be the solid region that is bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$. This shape is like a pyramid whose faces are all triangles. Make a 3d sketch of D and then make a 2d sketch of the region in the xy-plane of its bottom face. Set up a triple integral to find its mass given that it has density $f(x, y, z)$.

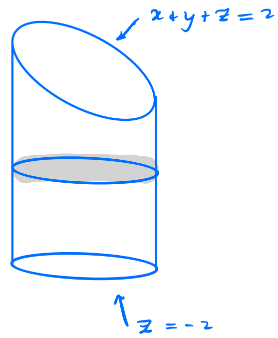


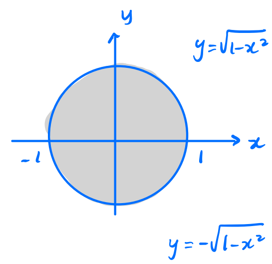


$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x, y, z)\, dz\, dy\, dx$$

Problem 3. Let D be the solid region that is given by a solid cylinder bounded on the sides by $x^2 + y^2 = 1$ whose bottom face is the plane $z = -2$ and whose top face is the plane $x + y + z = 2$. Make a 3d sketch of D and then make a 2d sketch of its cross section with the xy-plane. Set up a triple integral to find its mass given that it has density $f(x, y, z)$.

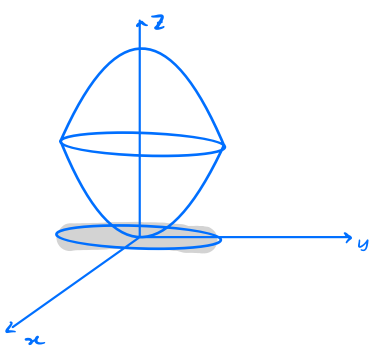


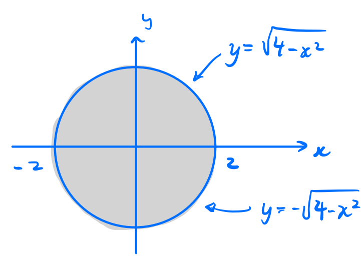




$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-2}^{2-x-y} 1 \, dz \, dy \, dx$$

Problem 4. Let D be the solid region that is bounded below by the paraboloid $z = x^2 + y^2$ and above by the parabolid $z = 8 - (x^2 + y^2)$. Make a 3d sketch of D. Find where the two paraboloids intersect and make a sketch in the xy-plane of the 2d region enclosed by their intersection. Set up a triple integral to find its mass given that it has density $f(x, y, z)$.





Intersection of a paraboloids:

$$x^2 + y^2 = 8 - (x^2 + y^2)$$

$\Rightarrow x^2 + y^2 = 4$ (this is the circle of points where they meet)

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-(x^2+y^2)} 1 \, dz \, dy \, dx$$