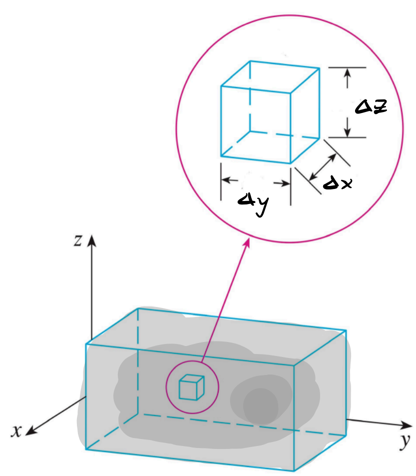
13.6 Triple Integrals

Suppose we have a region $D \subseteq \mathbb{R}^3$ which represents a solid whose mass is not necessarily uniformly distributed. For example the Earth's core is more dense than the mantle or crust. We could model the density by a function $f(x,y,z)$ which gives the mass per unit volume at the point (x,y,z) in D.

Suppose $D = [a,b] \times [c,d] \times [p,q] = \{(x,y,z) : a \le x \le b, c \le y \le d, p \le z \le q\}$

In the figure we could imagine darker areas as representing greater values of $f(x,y,z)$ (i.e. more dense regions)



If we subdivide D into $m \times n \times \ell$ sub-boxes, each with volume

$$\Delta V = \Delta x \, \Delta y \, \Delta z.$$

And in each sub-box D_{ijk}, we pick a point (x_i, y_j, z_k). The mass of D_{ijk} is then approximately

$$f(x_i, y_j, z_k)\,\Delta V$$

Then the total mass will be approximately

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} f(x_i, y_j, z_k)\,\Delta V$$

and we'll define the triple integral as

$$\iiint_D f(x,y,z) = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} f(x_i, y_j, z_k)\,\Delta V$$

$$= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx$$

The triple integral then represents the mass of D.

Note When $f(x,y,z) = 1$, $\iiint_D 1\,dV$ represents the volume of D.

Example Let $D = [0,2] \times [0,1] \times [0,3]$ and let $f(x,y,z) = 1 + xyz$ be the density of D. Find the mass of D.

$$\text{mass} = \int_0^2 \int_0^1 \left(\int_0^3 (1+xyz)\, dz \right) dy\, dx$$

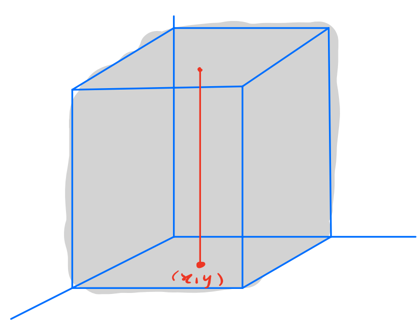
$$= \int_0^2 \int_0^1 \left(z + \frac{1}{2} xyz^2 \Big|_0^3 \right) dy\, dx$$

$$= \int_0^2 \int_0^1 \left(3 + \frac{9}{2} xy\right) dy\, dx$$

$$= \int_0^2 \left(3y + \frac{9}{4} xy^2 \Big|_0^1 \right) dx$$

$$= \int_0^2 \left(3 + \frac{9}{4} x\right)$$

$$= 3x + \frac{9}{8} x^2 \Big|_0^2 = 6 + \frac{9}{2} = \frac{21}{2}$$



$$\int_0^3 (1+xyz)\, dz = 3 + \frac{9}{2} xy$$

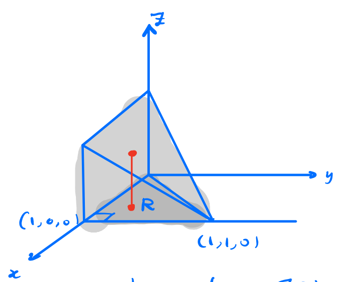
represents the mass of of the vertical line segment over an arbitrary point $(x,y) \in [0,2] \times [0,1]$

Example Let D be the solid region shown below.

It is the solid with upper and lower bounds $z=0$ and $z=1-y$.

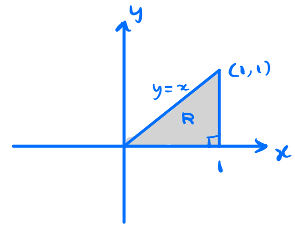
Its sides are the vertical planes $y=0$, $x=1$, and $y=x$.

Set up a triple integral for its mass, given density $f(x,y,z)$.



top surface: $z=1-y$

bottom: $z=0$



$$\underbrace{\int_{x=0}^{x=1}\int_{y=0}^{y=x}}_{\text{double integral over } R}\underbrace{\left(\int_{z=0}^{z=1-y} f(x,y,z)\,dz\right)}_{\text{represents mass of vertical line segment over arbitrary } (x,y) \text{ in } R}dy\,dx$$

Guidelines for triple integrals

1. outer limits should be constants
2. middle limits can only depend on outer variable
3. inner limits can depend on outer and middle variables

Let's generalize this. We'll think of

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{z=G_1(x,y)}^{z=G_2(x,y)} f(x,y,z)\, dz \right) dA$$

in the following way:

① D is a solid bounded vertically by

$z = G_1(x,y)$ (lower bound) and $z = G_2(x,y)$ (upper bound)

and $\displaystyle\int_{G_1(x,y)}^{G_2(x,y)} f(x,y,z)\, dz$ represents the mass of a

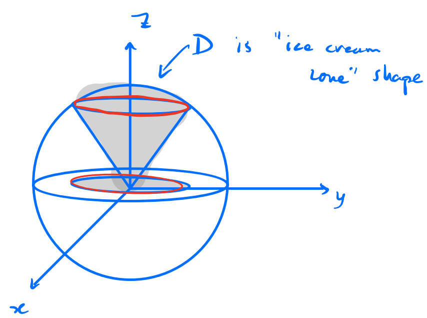
vertical line over an arbitrary point (x,y) in R.

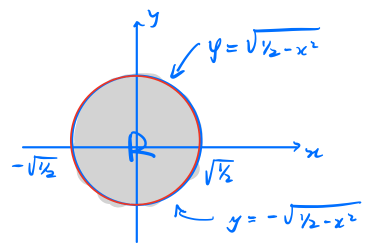
② R is the region in the xy-plane which is the

projection of D onto the xy-plane

Example Let D be the solid region that is bounded between the cone $z = \sqrt{x^2+y^2}$ and the hemisphere $z = \sqrt{1-x^2-y^2}$.

Set up a triple integral for its volume.





R is projection of D onto xy-plane

Intersection circle

$$\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2}$$

$$\Rightarrow \quad x^2+y^2 = 1 - x^2 - y^2$$

$$\Rightarrow \quad x^2+y^2 = \tfrac{1}{2} \quad (\text{circle of radius } \sqrt{1/2}\,)$$

$$\iiint_D 1 \, dV = \iint_R \left(\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{1-x^2-y^2}} 1 \, dz \right) dA$$

$$= \int_{x=-\sqrt{1/2}}^{x=\sqrt{1/2}} \int_{y=-\sqrt{1/2-x^2}}^{y=\sqrt{1/2-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{1-x^2-y^2}} 1 \, dz\,dy\,dx$$

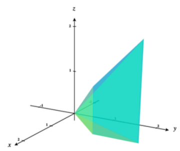
Guide to triple integrals with z inside limits

1. Sketch D

2. Collapse D onto region R in xy-plane and sketch R

3. Outer and middle limits are double integral over R, inner limits are top and bottom surfaces of D

Problem 1. Consider the triple integral $\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dzdydx$. This is a triple integral over the region D shown below.



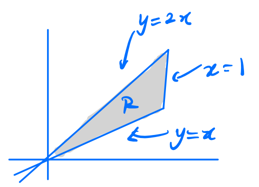
a. Identify the lower and upper bounding surfaces of D.

b. Sketch the region R given by projecting D onto the xy-plane.

c. Compute the integral.

(a) $z = 0\ ,\ z = y$

(b)



(c) $\int_0^1 \int_x^{2x} \left(xyz^2 \Big|_0^y \right) dy\,dx$

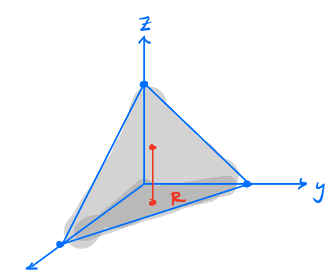
$= \int_0^1 \int_x^{2x} xy^3\, dy\,dx$

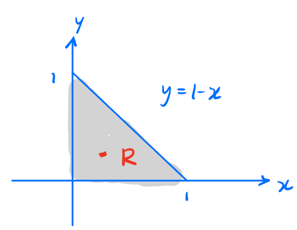
$= \int_0^1 \left(\frac{1}{4} x y^4 \Big|_x^{2x} \right) dx$

$= \int_0^1 \frac{1}{4} x \left(16x^4 - x^4 \right) dx$

$= \frac{15}{4} \int_0^1 x^5\, dx = \boxed{\frac{15}{24}}$

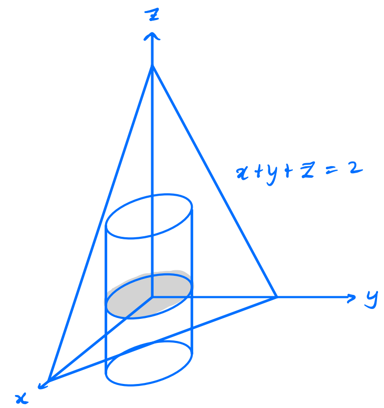
Problem 2. Let D be the solid region that is bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$. This shape is like a pyramid whose faces are all triangles. Make a 3d sketch of D and then make a 2d sketch of the region in the xy-plane of its bottom face. Set up a triple integral to find its mass given that it has density $f(x, y, z)$.

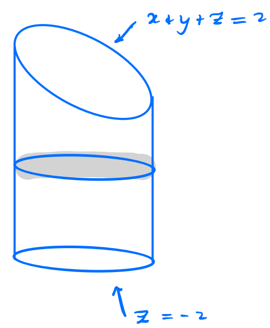


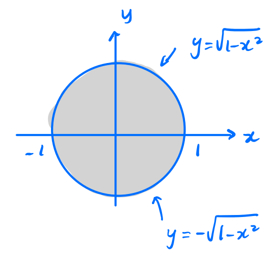


$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\, dz\,dy\,dx$$

Problem 3. Let D be the solid region that is given by a solid cylinder bounded on the sides by $x^2 + y^2 = 1$ whose bottom face is the plane $z = -2$ and whose top face is the plane $x + y + z = 2$. Make a 3d sketch of D and then make a 2d sketch of its cross section with the xy-plane. Set up a triple integral to find its mass given that it has density $f(x, y, z)$.

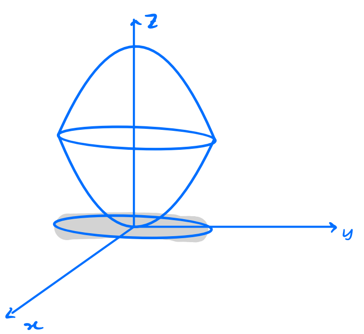


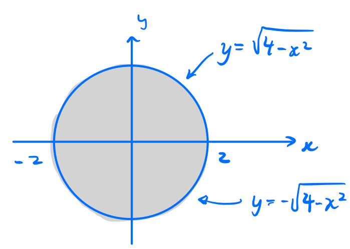




$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-2}^{2-x-y} 1\, dz\, dy\, dx$$

Problem 4. Let D be the solid region that is bounded below by the paraboloid $z = x^2 + y^2$ and above by the parabolid $z = 8 - (x^2 + y^2)$. Make a 3d sketch of D. Find where the two paraboloids intersect and make a sketch in the xy-plane of the 2d region enclosed by their intersection. Set up a triple integral to find its mass given that it has density $f(x, y, z)$.





Intersection of a paraboloids:

$$x^2 + y^2 = 8 - (x^2 + y^2)$$

$\Rightarrow x^2 + y^2 = 4$ (this is the circle of points where they meet)

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-(x^2+y^2)} 1\, dz\, dy\, dx$$