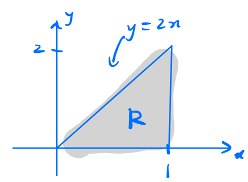
13.6 More triple integrals

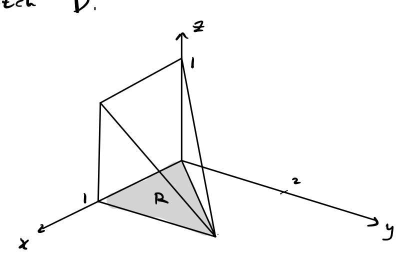
Goal Set up triple integrals using other orders of integration (not necessarily using z for inside limits of integration)

Example Consider the integral $\int_0^1 \int_0^{2x} \int_0^{1-y/2} f(x,y,z)\, dz\, dy\, dx$.

① Sketch the projection R of D onto the xy-plane.



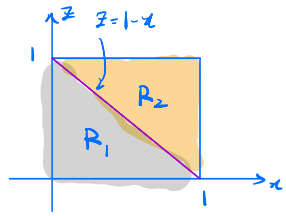
② Sketch D.



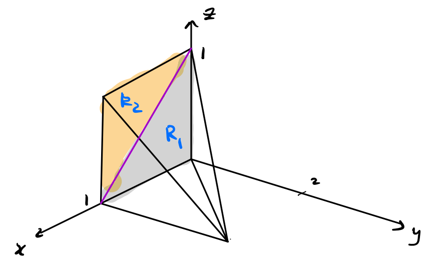
$z = 1 - y/2$ is top surface

$z = 0$ is bottom surface

③ Set up the triple integral with $dA = dy\,dz\,dx$



projection of D onto xz-plane

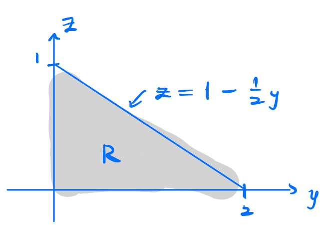


$R_1 \begin{cases} y \geq 0 & \text{is left surface} \\ y = 2x & \text{is right surface} \end{cases}$

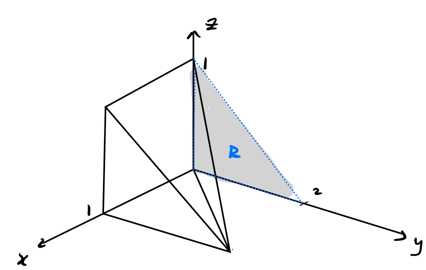
$R_2 \begin{cases} y = 0 & \text{is left surface} \\ y = 1 - 2z & \text{is right surface} \end{cases}$

$$\int_{x=0}^{x=1} \int_{z=0}^{z=1-x} \int_{y=0}^{y=2x} f(x,y,z)\,dy\,dz\,dx + \int_{x=0}^{x=1} \int_{z=1-x}^{z=1} \int_{y=0}^{y=1-2z} f(x,y,z)\,dy\,dz\,dx$$

④ Set up with $dA = dx\,dz\,dy$



projection of D onto yz-plane



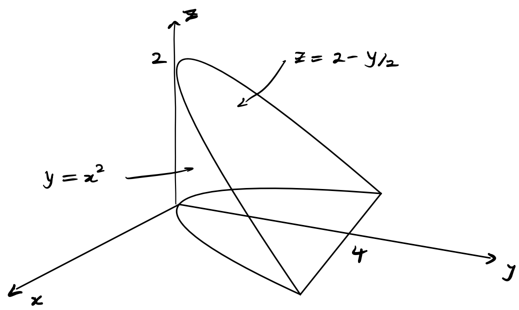
$x = \frac{1}{2}y$ is back surface

$x = 1$ is front surface

$$\int_{y=0}^{y=2} \int_{z=0}^{z=1-\frac{y}{2}} \int_{x=\frac{1}{2}y}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

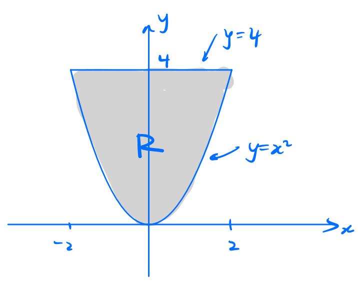
Example Let D be the solid bounded by the surfaces

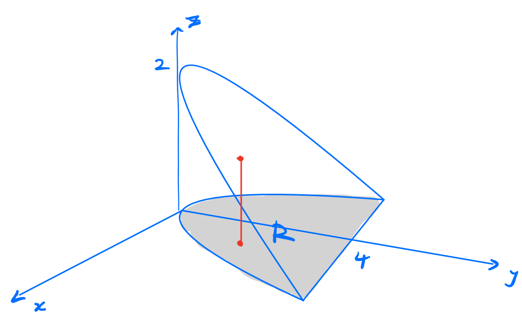
$y = x^2$, $z = 0$, and $z = 2 - y/2$.



Set up $\displaystyle\iiint_D f(x,y,z)\, dV$ using

① $dV = dz\, dy\, dx$



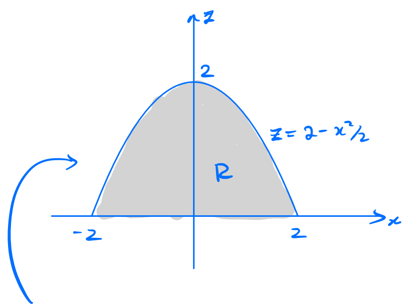


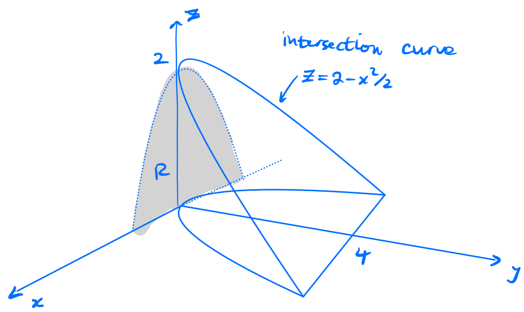
bottom surface $z=0$

top surface $z = 2 - y/2$

$$\underbrace{\int_{x=-2}^{x=2}\int_{y=x^2}^{y=4}}_{\substack{\text{double integral}\\ \text{over } R}} \underbrace{\left(\int_{z=0}^{z=2-y/2} f(x,y,z)\, dz\right)}_{\substack{\text{mass of vertical line in } D\\ \text{through arbitrary } (x,y) \text{ in } R}} dy\, dx$$

(2) $dV = dy\,dz\,dx$





intersection of $y = x^2$, $z = 2 - y/2$

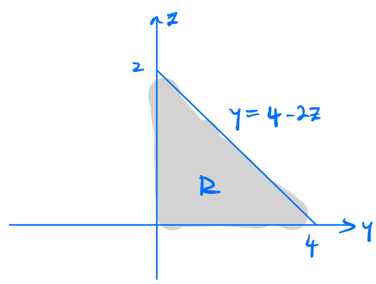
is $z = 2 - x^2/2$

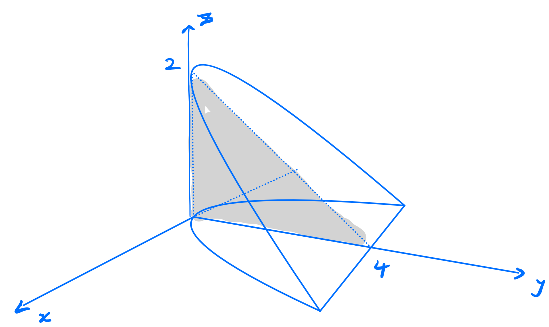
left surface $y = x^2$

right surface $y = 4 - 2z$

$$\underbrace{\int_{x=-2}^{x=2}\int_{z=0}^{z=2-x^2/2}}_{\text{double integral over } R} \underbrace{\left(\int_{y=x^2}^{y=4-2z} f(x,y,z)\,dy\right)}_{\text{mass of horiz. line in } D \text{ through arbitrary } (x,z) \text{ in } R} dz\,dx$$

(3) $dV = dx\,dy\,dz$

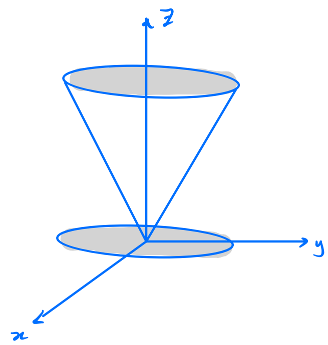


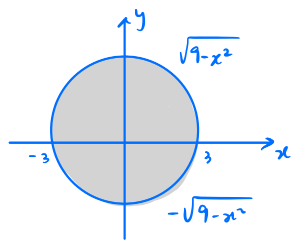


back surface $x = -\sqrt{y}$
front surface $x = \sqrt{y}$

$$\underbrace{\int_{z=0}^{z=2}\int_{y=0}^{y=4-2z}}_{\text{double integral over } R} \left(\underbrace{\int_{x=-\sqrt{y}}^{x=\sqrt{y}} f(x,y,z)\,dx}_{\text{mass of horiz. line in D through arbitrary } (y,z) \text{ in } R} \right) dy\,dz$$

Problem 1. Let D be the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 3$. Make a sketch of D and set up the triple integral $\iiint_D f(x, y, z)\,dV$ in two ways: using (1) $dV = dz\,dy\,dx$ and (2) $dV = dx\,dy\,dz$.



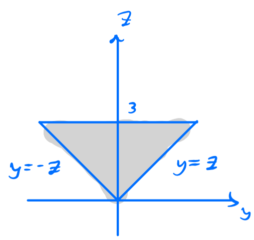


collapse onto xy-plane

intersection :

$$\begin{cases} z = 3 \\ z = \sqrt{x^2+y^2} \end{cases} \Rightarrow 9 = x^2 + y^2$$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} f(x,y,z)\,dz\,dy\,dx$$

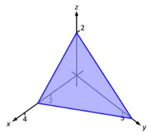


collapse onto yz-plane

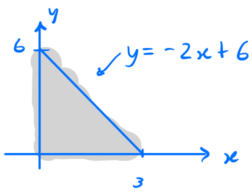
$$\int_{z=0}^{z=3} \int_{y=-z}^{y=z} \int_{x=-\sqrt{z^2-y^2}}^{x=\sqrt{z^2-y^2}} f(x,y,z)\,dx\,dy\,dz$$

(back and front surfaces found by solving $z = \sqrt{x^2+y^2}$ for x)

Problem 2. Let D be the region bounded by $x = 0, y = 0, z = 0$, and $z = 2 - y/3 - 2x/3$ (shown below). Set up the triple integral $\iiint_D dV$ using the following choices of dV: (1) $dzdydx$, (2) $dxdydz$, and (3) $dxdzdy$.

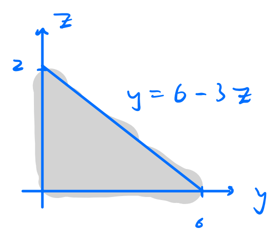


(1)



$$\int_0^3 \int_0^{-2x+6} \int_0^{2-y/3-2x/3} 1 \, dz \, dy \, dx$$

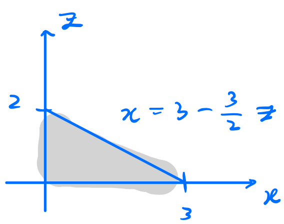
(2)



$$z = 2 - y/3 - 2x/3$$
$$\Rightarrow 3z = 6 - y - 2x$$
$$\Rightarrow x = 3 - \frac{1}{2}y - \frac{3}{2}z$$

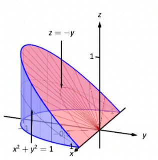
$$\int_0^2 \int_0^{6-3z} \int_0^{3-\frac{1}{2}y-\frac{3}{2}z} 1 \, dx \, dy \, dz$$

(3)

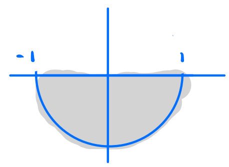


$$\int_0^2 \int_0^{3-\frac{3}{2}z} \int_0^{6-2x-3z} 1 \, dy \, dx \, dz$$

Problem 3. Let D be the region shown below. Set up the triple integral $\iiint_D dV$ using the following choices of dV: (1) $dz\,dy\,dx$, (2) $dx\,dy\,dz$, and (3) $dy\,dx\,dz$.

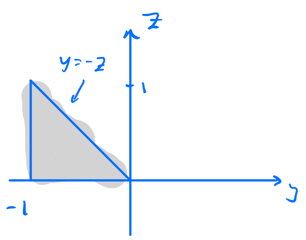


(1)



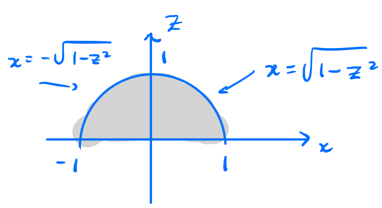
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} 1\,dz\,dy\,dx$$

(2)



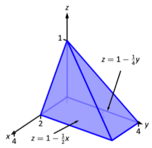
$$\int_{0}^{1}\int_{-1}^{-z}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1\,dx\,dy\,dz$$

(3)

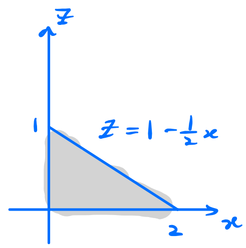


$$\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-x^2}}^{-z} 1\,dy\,dx\,dz$$

Problem 4. Let D be the region shown below. Set up the triple integral $\iiint_D dV$ using the following choices of dV: (1) $dydzdx$ and (2) $dzdydx$. *Hint: the second setup is trickier than previous problems; you'll need to write it as a sum of two triple integrals.*

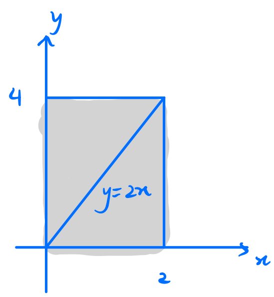


(1)



$$\int_{x=0}^{x=2}\int_{z=0}^{z=1-\frac{1}{2}x}\int_{y=0}^{y=4(1-z)} 1\,dy\,dz\,dx$$

(2)



$$\int_{x=0}^{x=2}\int_{y=2x}^{y=4}\int_{z=0}^{z=1-\frac{1}{4}y} dz\,dy\,dx$$

$$+\int_{x=0}^{x=2}\int_{y=0}^{y=2x}\int_{z=0}^{z=1-\frac{1}{2}x} dz\,dy\,dx$$