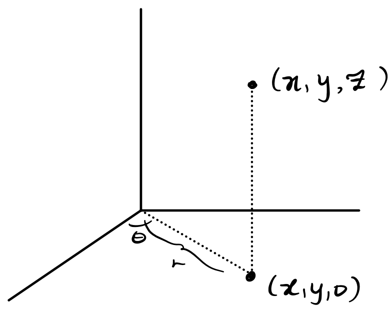
13.7 Cylindrical coordinates

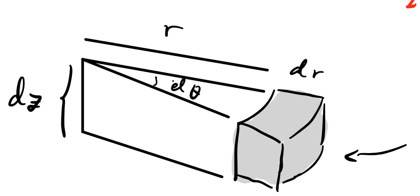


Given a point (x, y, z) in $\mathbb{R}^3$ in Cartesian coordinates it's possible to express it in a new coordinate system called cylindrical coordinates where (r, θ, z) is the point and r, θ are the polar description of x, y.

$x^2 + y^2 = r^2$ $\qquad$ $x = r\cos\theta$

$\tan\theta = \frac{y}{x}$ $\qquad$ $y = r\sin\theta$

$z = z$

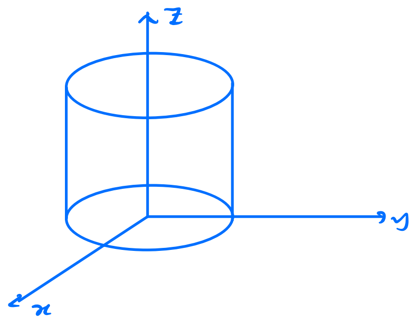


volume of infinitesimal region

$= dV = r\,dz\,dr\,d\theta$

Example What graphs do the following cylindrical equations represent?

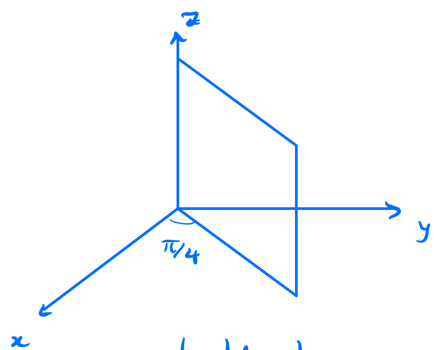
① $r = 3$



hollow cylinder

$x^2 + y^2 = 9$

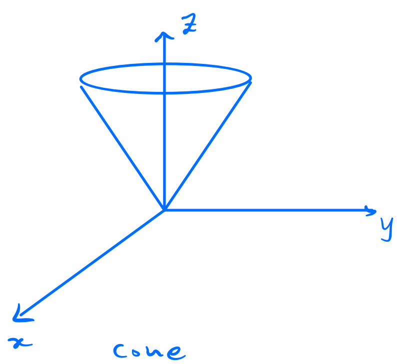
② $\theta = \frac{\pi}{4}$



half plane

$y = x$

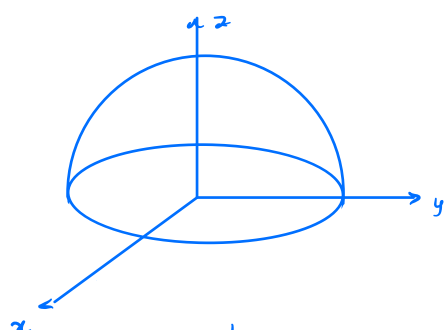
③ $z = r$



cone

$z = \sqrt{x^2 + y^2}$

④ $z = \sqrt{1 - r^2}$



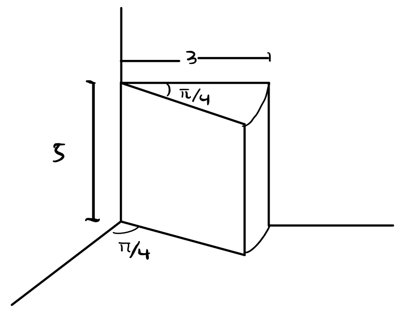
upper hemisphere

$z = \sqrt{1 - x^2 - y^2}$

Example Set up a triple integral for $\iiint_D 1 \, dV$

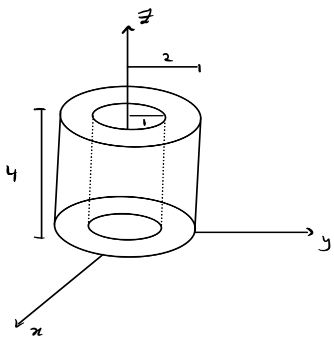
when D is given by the solid regions below.

(1)



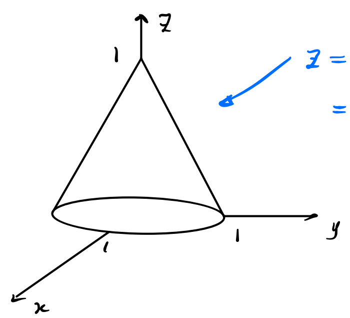
$$\int_{\pi/4}^{\pi/2} \int_0^3 \int_0^5 1 \, r \, dz \, dr \, d\theta$$

(2)



$$\int_0^{2\pi} \int_1^2 \int_0^4 1 \, r \, dz \, dr \, d\theta$$

(3)

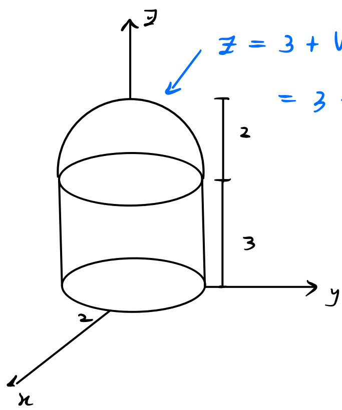


$z = 1 - \sqrt{x^2 + y^2}$
$= 1 - r$

$$\int_0^{2\pi} \int_0^1 \int_0^{1-r} 1 \, r \, dz \, dr \, d\theta$$

(4)

$z = 3 + \sqrt{4 - x^2 - y^2}$
$= 3 + \sqrt{4 - r^2}$



$$\int_0^{2\pi} \int_0^2 \int_0^{3+\sqrt{4-r^2}} 1 \, r \, dz \, dr \, d\theta$$

Problem 1. Set up a triple integral in cylindrical coordinates for the volume of each solid region described below. Begin by sketching the region.

a. Bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 25$ and the planes $z = 0$ and $z = 6$.

b. Bounded by the half-cylinder $x^2+y^2 = 9$ where $x \leq 0$ with height 4 and base on the xy-plane.

c. Bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by unit sphere.

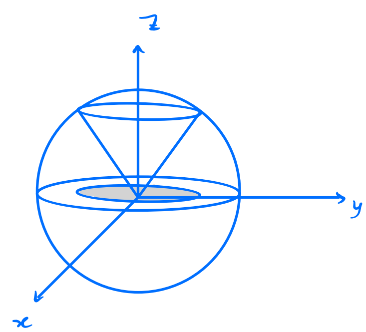
d. Bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 2$.

e. The solid cylinder bounded by $x^2 + y^2 = 1$ and $z = 0, z = 1$ but with the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 1$ removed.

(a) $$\int_0^{2\pi} \int_1^5 \int_0^6 r \, dz \, dr \, d\theta$$

(b) $$\int_{\pi/2}^{3\pi/2} \int_0^3 \int_0^4 r \, dz \, dr \, d\theta$$

(c)



Intersection:

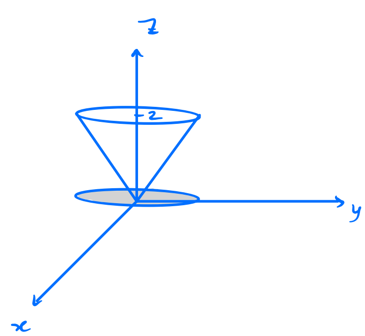
$$\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2}$$

$$\Rightarrow x^2+y^2 = 1 - x^2 - y^2$$

$$\Rightarrow x^2+y^2 = \frac{1}{2} \quad (\text{circle of radius } \sqrt{1/2})$$

$$\int_0^{2\pi} \int_0^{\sqrt{1/2}} \int_r^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

(d)

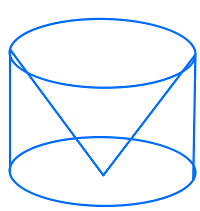


$$\sqrt{x^2+y^2} = 2$$

$$\Rightarrow x^2+y^2 = 4 \quad (\text{circle of radius } 2)$$

$$\int_0^{2\pi} \int_0^2 \int_r^2 r \, dz \, dr \, d\theta$$

(e)



$$\int_0^{2\pi} \int_0^1 \int_0^r r \, dz \, dr \, d\theta$$

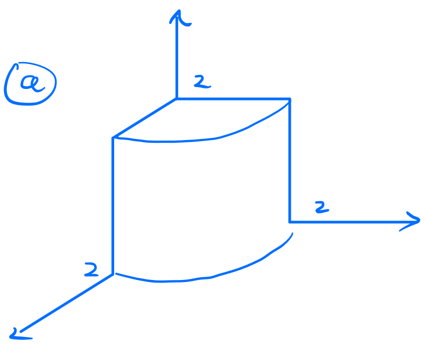
Problem 2. A triple integral in cylindrical coordinates is given. Describe the region of integration in words and with a sketch.

a. $\int_0^{\pi/2} \int_0^2 \int_0^2 r \, dz dr d\theta$

b. $\int_0^{\pi} \int_0^1 \int_0^{1-r} r \, dz dr d\theta$

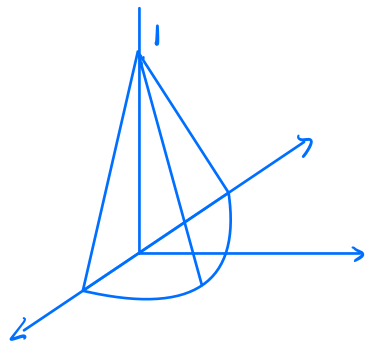
c. $\int_0^{2\pi} \int_0^1 \int_0^{2-r} r \, dz dr d\theta$

(a)



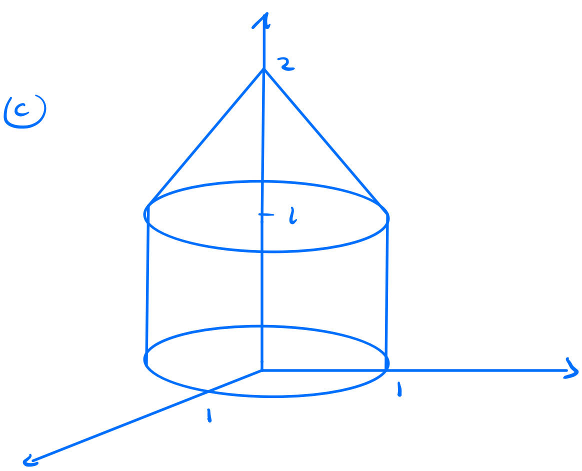
solid quarter cylinder

(b) $z = 1 - r = 1 - \sqrt{x^2+y^2}$



solid half cone

(c)



solid cylinder with solid cone on top