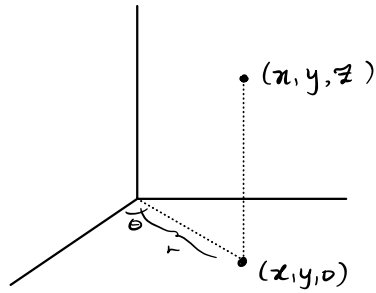


13.7 Cylindrical coordinates



Given a point (x, y, z) in \mathbb{R}^3 in Cartesian coordinates it's possible to express it in a new coordinate system called cylindrical coordinates where (r, θ, z) is the point and r, θ are the polar description of x, y .

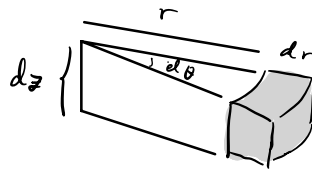
$$x^2 + y^2 = r^2$$

$$\tan \theta = \frac{y}{x}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$



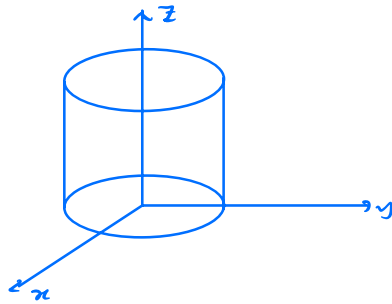
← volume of infinitesimal region

$$= dV = dz dA$$

$$= r dz dr d\theta$$

Example What graphs do the following cylindrical equations represent?

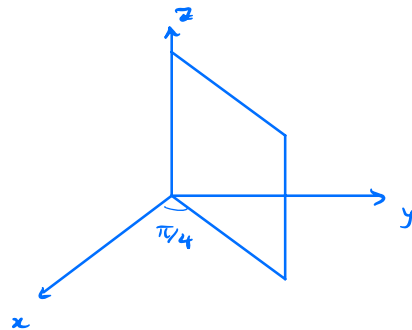
① $r = 3$



hollow cylinder

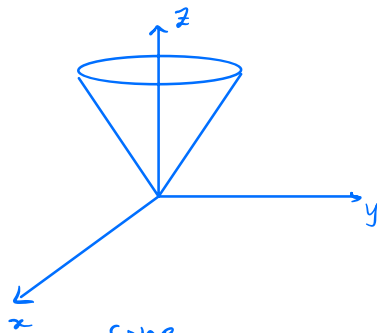
$$x^2 + y^2 = 9$$

② $\theta = \frac{\pi}{4}$



plane $y = x$

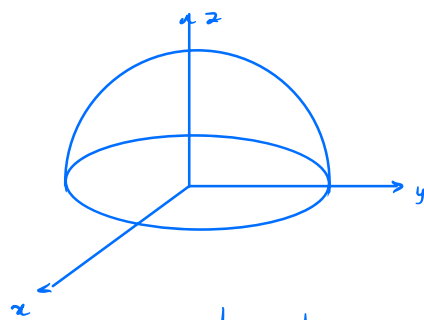
③ $z = r$



cone

$$z = \sqrt{x^2 + y^2}$$

④ $z = \sqrt{1 - r^2}$



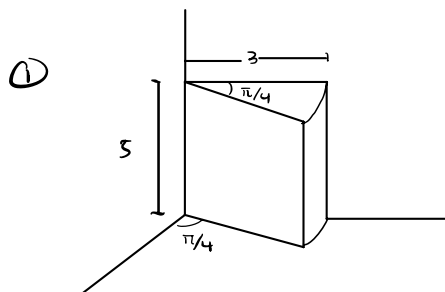
upper hemisphere

$$z = \sqrt{1 - x^2 - y^2}$$

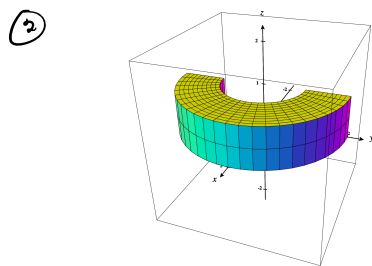
Example Set up a triple integral for $\iiint_D 1 dV$

when D is given by the solid regions below.

piece of cake

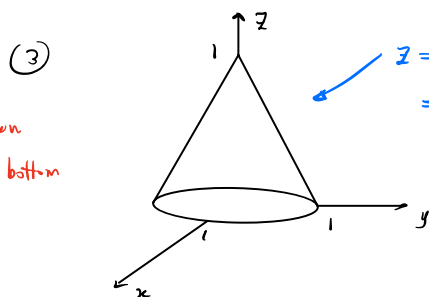


$$\int_{\pi/4}^{\pi/2} \int_0^3 \int_0^5 1 r dz dr d\theta$$



$$\int_{-\pi/2}^{\pi/2} \int_1^2 \int_0^4 1 r dz dr d\theta$$

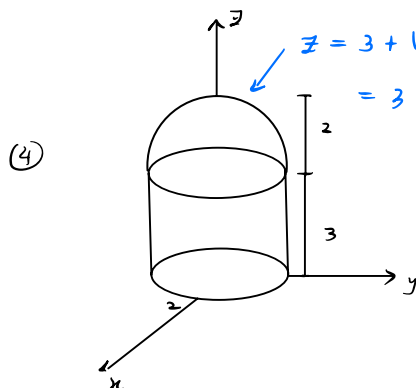
upside down cone, flat bottom



$$z = 1 - \sqrt{x^2 + y^2} \\ = 1 - r$$

$$\int_0^{2\pi} \int_0^1 \int_0^{1-r} 1 r dz dr d\theta$$

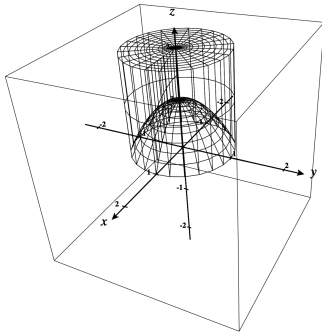
bird cage



$$z = 3 + \sqrt{4 - x^2 - y^2} \\ = 3 + \sqrt{4 - r^2}$$

$$\int_0^{2\pi} \int_0^2 \int_0^{3 + \sqrt{4 - r^2}} 1 r dz dr d\theta$$

Example Consider the solid region within the cylinder of height 2, radius 1 which is between $z = 1 - x^2 - y^2$ and $z = 2$. Suppose the solid has



density at each point which is proportional to the distance from the point to the z -axis.

Find the total mass of the solid in terms of the proportionality constant k .

$$\text{density: } f(x, y, z) = k\sqrt{x^2 + y^2} = kr$$

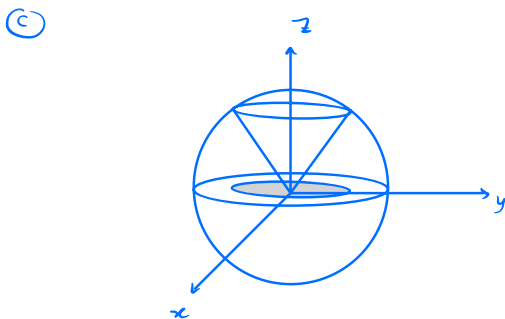
$$\begin{aligned} \text{mass} &= \iiint_D f(x, y, z) dV \\ &= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^2 k r^2 dz dr d\theta \\ &= k \int_0^{2\pi} \int_0^1 r^2 (z \Big|_{1-r^2}^2) dr d\theta \\ &= k \int_0^{2\pi} \int_0^1 r^2 (1+r^2) dr d\theta \\ &= k \int_0^{2\pi} \left(\frac{1}{3} r^3 + \frac{1}{5} r^5 \Big|_0^1 \right) d\theta \\ &= \frac{8k}{15} \int_0^{2\pi} d\theta \\ &= \frac{16\pi}{15} \end{aligned}$$

Problem 1. Set up a triple integral in cylindrical coordinates for the volume of each solid region described below. Begin by sketching the region.

- Bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 25$ and the planes $z = 0$ and $z = 6$.
- Bounded by the half-cylinder $x^2 + y^2 = 9$ where $x \leq 0$ with height 4 and base on the xy -plane.
- Bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by unit sphere.
- Bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 2$.
- The solid cylinder bounded by $x^2 + y^2 = 1$ and $z = 0, z = 1$ but with the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 1$ removed.

(a)
$$\int_0^{2\pi} \int_1^5 \int_0^6 r \, dz \, dr \, d\theta$$

(b)
$$\int_{\pi/2}^{3\pi/2} \int_0^3 \int_0^4 r \, dz \, dr \, d\theta$$



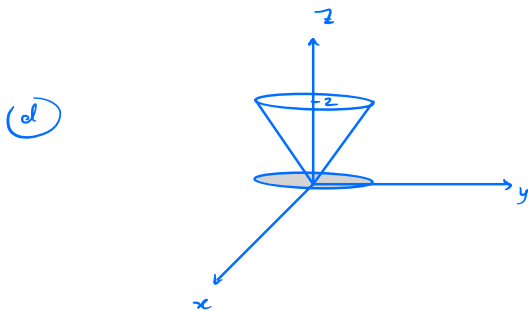
Intersection:

$$\sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2}$$

$$\Rightarrow x^2 + y^2 = 1 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 = \frac{1}{2} \quad (\text{circle of radius } \sqrt{1/2})$$

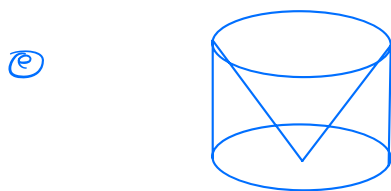
$$\int_0^{2\pi} \int_0^{\sqrt{1/2}} \int_r^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$



$$\sqrt{x^2 + y^2} = z$$

$$\Rightarrow x^2 + y^2 = 4 \quad (\text{circle of radius } 2)$$

$$\int_0^{2\pi} \int_0^2 \int_r^2 r \, dz \, dr \, d\theta$$



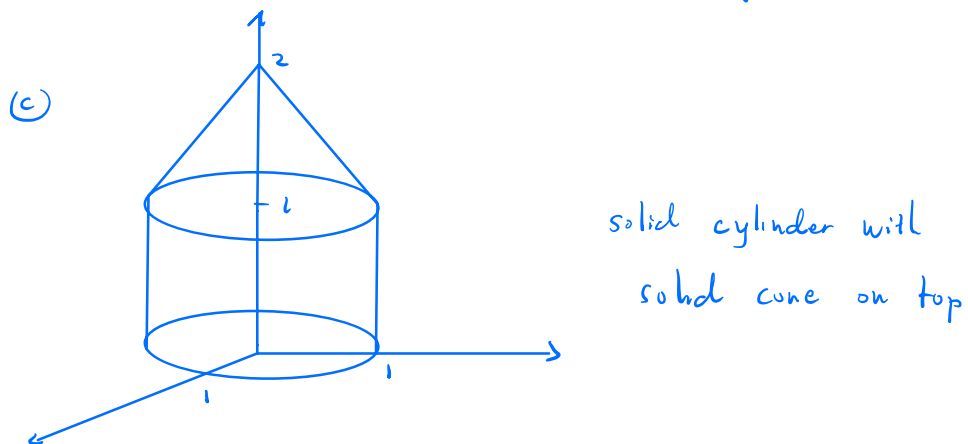
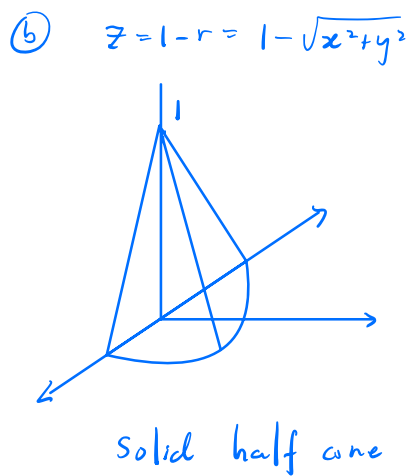
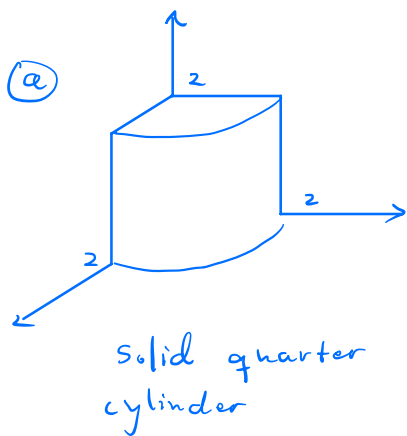
$$\int_0^{2\pi} \int_0^1 \int_0^r r \, dz \, dr \, d\theta$$

Problem 2. A triple integral in cylindrical coordinates is given. Describe the region of integration in words and with a sketch.

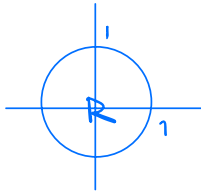
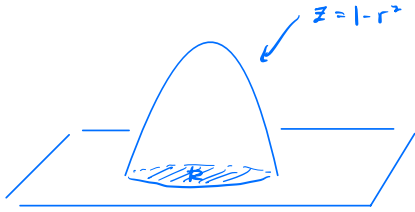
a. $\int_0^{\pi/2} \int_0^2 \int_0^2 r \, dz \, dr \, d\theta$

b. $\int_0^{\pi} \int_0^1 \int_0^{1-r} r \, dz \, dr \, d\theta$

c. $\int_0^{2\pi} \int_0^1 \int_0^{2-r} r \, dz \, dr \, d\theta$



Problem 3. A solid is bounded from above by the paraboloid $z = 1 - x^2 - y^2$ and from below by the xy -plane. It has density $f(x, y, z) = 15 - 3z$, meaning it is heavier near the base and lighter near the top. Use cylindrical coordinates to find the mass of the solid.



$$\begin{aligned}
 & \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (15-3z) r dz dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 r \left(15z - \frac{3}{2} z^2 \Big|_0^{1-r^2} \right) dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 r (15(1-r^2) - \frac{3}{2} (1-r^2)^2) dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 r (15 - 15r^2 - \frac{3}{2} (1 - 2r^2 + r^4)) dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 r \left(\frac{27}{2} - 12r^2 - \frac{3}{2} r^4 \right) dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left(\frac{27}{2} r - 12r^3 - \frac{3}{2} r^5 \right) dr d\theta \\
 &= \int_0^{2\pi} \left(\frac{27}{4} r^2 - 3r^4 - \frac{1}{4} r^6 \Big|_0^1 \right) d\theta \\
 &= \int_0^{2\pi} \left(\frac{27}{4} - 3 - \frac{1}{4} \right) d\theta \\
 &= \int_0^{2\pi} \frac{7}{2} d\theta \\
 &= 7\pi
 \end{aligned}$$