Problem 1. Consider the function $f(x,y) = 4 - (x-1)^2 - (y-2)^2$.

a. What are the domain and range of f?

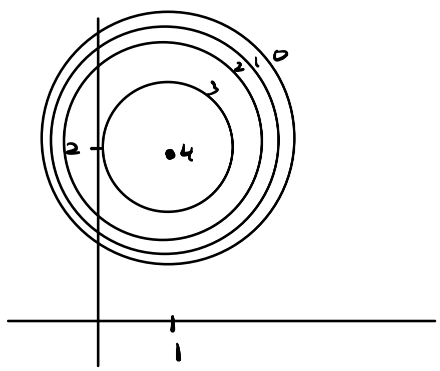
b. Make a contour diagram using the level curves $f(x,y) = c$ where $c = 0, 1, 2, 3, 4$.

c. Explain why a contour diagram of f cannot have level curves where $c > 4$.

d. Sketch the graph of f.

(a) domain is $\mathbb{R}^2$ and range is $(-\infty, 4]$

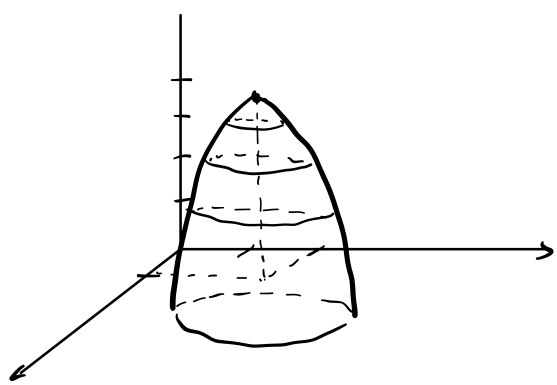
(b)



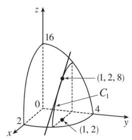
c	curve
0	$(x-1)^2 + (y-2)^2 = 4$
1	$(x-1)^2 + (y-2)^2 = 3$
2	$(x-1)^2 + (y-2)^2 = 2$
3	$(x-1)^2 + (y-2)^2 = 1$
4	$(x-1)^2 + (y-2)^2 = 0$

(c) The c-values must be in the range of f

(d)



Problem 2. Consider the function f whose graph is shown below.



a. Find the value of $f(1, 2)$.

b. Which of the following is an equation for the curve C_1?

1. $z = f(1, 2)$
2. $z = f(x, 2)$
3. $z = f(1, y)$

c. Find the sign (positive, negative, or zero) of the following quantities:

1. $f_x(1, 2)$
2. $f_y(1, 2)$
3. $f_{xx}(1, 2)$
4. $f_{yy}(1, 2)$
5. $D_{\mathbf{u}} f(1, 2)$ where $\mathbf{u} = \langle -1, -1 \rangle$
6. $\|\nabla f(1, 2)\| - D_{-i} f(1, 2)$

(a) 8

(b) 2

(c) (1) neg

(2) neg

(3) neg

(4) neg

(5) pos

(6) pos

Problem 3. For each of the following examples, find $D_{\mathbf{u}}f(P)$.

a. $f(x, y) = xy$, $P = (0, -2)$
   1. $\mathbf{u}$ in the direction of $\langle 1, 3\rangle$
   2. $\mathbf{u}$ in the direction of maximum rate of increase

b. $f(x, y) = e^x \sin y$, $P = (1, \pi/2)$
   1. $\mathbf{u}$ in the direction of $\langle -1, 1\rangle$
   2. $\mathbf{u}$ in the direction of maximum rate of decrease

c. $f(x, y) = xe^{-y}$ $P = (1, 0)$
   1. $\mathbf{u}$ in the direction of $\langle 0, 1\rangle$
   2. $\mathbf{u}$ in the direction given by rotating ∇f by $\pi/2$ counterclockwise

d. $f(x, y) = \sqrt{x^2 + 2y}$ at $(4, 10)$
   1. $\mathbf{u}$ in the direction of $\langle 2, 0\rangle$
   2. $\mathbf{u}$ in the direction given by rotating ∇f by $\pi/2$ clockwise

(a) $\nabla f(x,y) = \langle y, x\rangle \quad \nabla f(P) = \langle -2, 0\rangle$

(1) $\vec{u} = \frac{1}{\sqrt{10}}\langle 1, 3\rangle \quad D_{\vec{u}}f(0,-2) = \nabla f(0,-2)\cdot\vec{u}$

$= \frac{-2}{\sqrt{10}}$

(2) $\|\nabla f(P)\| = 2$

(b) $\nabla f(x,y) = \langle e^x \sin y, e^x \cos y\rangle, \quad \nabla f(P) = \langle e, 0\rangle$

(1) $\vec{u} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle \quad D_{\vec{u}} f(1, \pi/2) = \frac{-e}{\sqrt{2}}$

(2) $-\|\nabla f(P)\| = -e$

(c) $\nabla f(x,y) = \langle e^{-y}, -xe^{-y}\rangle \quad \nabla f(P) = \langle 1, -1\rangle$

(1) $\nabla f(P)\cdot\langle 0, 1\rangle = -1$

(2) 0

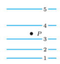
(d) $\nabla f(x,y) = \left\langle \frac{x}{\sqrt{x^2+2y}}, \frac{1}{\sqrt{x^2+2y}}\right\rangle$

$\nabla f(P) = \left\langle \frac{4}{\sqrt{16+20}}, \frac{1}{6}\right\rangle = \left\langle \frac{2}{3}, \frac{1}{6}\right\rangle$

(1) $\nabla f(P)\cdot\langle 1, 0\rangle = \frac{2}{3}$

(2) 0

Problem 4. The figure below shows a contour plot of $f(x, y)$. Use the contour plot to determine the sign (positive, negative, or zero) of $f_x(P), f_y(P), f_{xx}(P), f_{yy}(P), f_{xy}(P)$.



deriv.	sign
$f_x(P)$	0
$f_y(P)$	+
$f_{xx}(P)$	0
$f_{yy}(P)$	−
$f_{xy}(P)$	0

Problem 5. The table below gives values of the function $V = f(T, P)$ where V is the volume (in cubic feet) of one pound of steam at a temperature T (in degrees Fahrenheit) and pressure P (in pounds per square inch).

Temperature T (°F) \ Pressure P (lb/in^2)	20	22	24	26
480	27.85	25.31	23.19	21.39
500	28.46	25.86	23.69	21.86
520	29.06	26.41	24.20	22.33
540	29.66	26.95	24.70	22.79

a. Give the local linear approximation $L(t, p)$ of V near $t = 500$ and $p = 24$. Keep in mind the following question when approximating partial derivatives.

b. Estimate the volume of a pound of steam at temperature 505 degrees Fahrenheit and pressure 24.3 pounds per square inch.

(a) $L_T(500, 24) \approx \dfrac{\Delta V}{\Delta T} = \dfrac{24.20 - 23.69}{520 - 500} = 0.0255$

$L_P(500, 24) \approx \dfrac{\Delta V}{\Delta P} = \dfrac{21.86 - 23.69}{26 - 24} = -0.915$

$$L(T, P) = 23.69 + 0.0255(T - 500) - 0.915(P - 24)$$

(b) $L(505, 24.3) = 23.69 + 0.0255(505 - 500) - 0.915(24.3 - 24)$

$= 23.543$

Problem 6. Consider the following functions. Find their critical points and use the Second Derivative Test to classify them.

a. $f(x,y) = y^2 + xy + 3y + 2x + 3$

b. $f(x,y) = x^2 + 2y^2 - x^2 y$

(a)

$$\begin{cases} f_x = y + 2 = 0 \\ f_y = 2y + x + 3 = 0 \end{cases} \Rightarrow \quad y = -2, \quad x = -2y - 3 = 4 - 3 = 1$$

$f_{xx} = 0, \quad f_{yy} = 2, \quad f_{xy} = 1$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = -1$

critical point	D	f_{xx}	classification
(1, -2)	-1		saddle point

(b)

$$\begin{cases} f_x = 2x - 2xy = 0 & ① \\ f_y = 4y - x^2 = 0 & ② \end{cases}$$

② $\Rightarrow y = x^2/4$,

① $2x\left(1 - \frac{x^2}{4}\right) = 0$

$\Rightarrow 2x(4 - x^2) = 0$

$\Rightarrow 2x(2-x)(2+x) = 0$

$\Rightarrow x = -2, 0, 2$

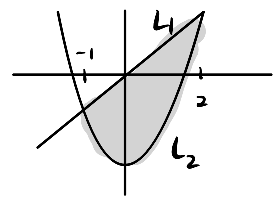
$f_{xx} = 2 - 2y, \quad f_{yy} = 4, \quad f_{xy} = -2x$

$D(x,y) = 4(2-2y) - (-2x)^2$

$= 4(2-2y) - 4x^2$

critical point	D	f_{xx}	classification
(-2, 1)	-16		saddle point
(0, 0)	8	2	local min
(2, 1)	-16		saddle point

Problem 7. Find the global minimum and maximum value, as well as the location of where they occur, of the function $f(x, y) = 3y - 2x$ constrained to the region bounded by $y = x^2 - 2$ and $y = x$.



$\nabla f = \langle -2, 3 \rangle$

no critical points

$x^2 - 2 = x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\underline{L_1}$ $y = x, \ -1 \leq x \leq 2$

$g_1(x) = f(x, x) = 3x - 2x = x$ on $[-1, 2]$

$g_1'(x) = 1$, no critical points

$g(-1) = -1$

$g(2) = 2$

$\underline{L_2}$ $y = x^2 - 2, \ -1 \leq x \leq 2$

$g_2(x) = f(x, x^2-2) = 3(x^2-2) - 2x$

$= 3x^2 - 2x - 6$

$g_2'(x) = 6x - 2$, $x = \frac{1}{3}$ is critical point

$g_2(-1) = -1$

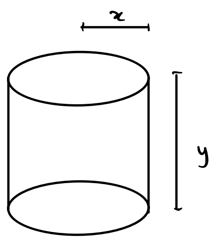
$g_2(\frac{1}{3}) = 3(\frac{1}{3})^2 - 2(\frac{1}{3}) - 6 = \frac{-19}{3}$

$g_2(2) = 2$

global min value is $\frac{-19}{3}$, occurs at $(\frac{1}{3}, \frac{-17}{9})$

global max value is 2, occurs at $(2, 2)$

Problem 8. A food manufacturing company is planning to make cans in the shape of circular cylinders with a volume of 10 cubic centimeters. In order to save on the cost of labels (on sides as well as tops and bottoms), they would like to minimize the surface area of the can. Use the method of Lagrange multipliers to find dimensions (radius x and height y) that fit these requirements.



minimize surface area

$$f(x,y) = 2\pi x^2 + 2\pi xy$$

with the constraint

$$g(x,y) = 10 \quad \text{where } g(x,y) = \pi x^2 y$$

and $x, y > 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 10 \end{cases} \Rightarrow \begin{cases} 4\pi x + 2\pi y = \lambda(2\pi xy) \\ 2\pi x = \lambda(\pi x^2) \\ \pi x^2 y = 10 \end{cases}$$

$$\Rightarrow \begin{cases} 2x + y = \lambda xy \\ 2 = \lambda x \Rightarrow x = \frac{2}{\lambda} \\ \pi x^2 y = 10 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{4}{\lambda} + y = 2y \Rightarrow y = \frac{4}{\lambda} \\ \pi\left(\frac{2}{\lambda}\right)^2 y = 10 \end{cases}$$

$$\Rightarrow \frac{16\pi}{\lambda^3} = 10 \Rightarrow \lambda = \left(\frac{8\pi}{5}\right)^{1/3}$$

$$x = \left(\frac{5}{\pi}\right)^{1/3}, \quad y = 2\left(\frac{5}{\pi}\right)^{1/3}$$

Problem 9. Let $f(x, y) = x^2 + 4y^2 - 2x + 8y$. Use the method of Lagrange Multipliers to determine the extreme values of f under the constraint that $x + 2y = 7$ and $x, y \geq 0$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ x + 2y = 7 \end{cases} \Rightarrow \begin{cases} 2x - 2 = \lambda & (1) \\ 8y + 8 = 2\lambda & (2) \\ x + 2y = 7 & (3) \end{cases}$$

$(1), (2) \Rightarrow 2x - 2 = 4y + 4,$

$x = 2y + 3,$

$(3) \Rightarrow 2y + 3 + 2y = 7, \; y = 1, \; x = 5$

$f(5, 1) = 25 + 4 - 10 + 8 = 27$, min value

$f(7, 0) = 49 - 14 = 35$

$f(0, \frac{7}{2}) = 4\left(\frac{7}{2}\right)^2 + 8\left(\frac{7}{2}\right) = 49 + 28 = 77$, max value