Problem 1. For each of the following examples, find $D_{\mathbf{u}}f(P)$.

a. $f(x, y) = xy$, $P = (0, -2)$
   1. $\mathbf{u}$ in the direction of $\langle 1, 3\rangle$
   2. $\mathbf{u}$ in the direction of maximum rate of change

b. $f(x, y) = e^x \sin y$, $P = (1, \pi/2)$
   1. $\mathbf{u}$ in the direction of $\langle -1, 1\rangle$
   2. $\mathbf{u}$ in the direction of minimum rate of change

c. $f(x, y) = xe^{-y}$ $P = (1, 0)$
   1. $\mathbf{u}$ in the direction of $\langle 0, 1\rangle$
   2. $\mathbf{u}$ in the direction given by rotating ∇f by $\pi/2$ counterclockwise

d. $f(x, y) = \sqrt{x^2 + 2y}$ at $(4, 10)$
   1. $\mathbf{u}$ in the direction of $\langle 2, 0\rangle$
   2. $\mathbf{u}$ in the direction given by rotating ∇f by $\pi/2$ clockwise

(a) $\nabla f(x,y) = \langle y, x\rangle$ $\nabla f(P) = \langle -2, 0\rangle$

① $\vec{u} = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$ $D_{\vec{u}}f(0,-2) = \nabla f(0,-2)\cdot\vec{u}$

$= \frac{-2}{\sqrt{10}}$

② $\|\nabla f(P)\| = 2$

(b) $\nabla f(x,y) = \langle e^x \sin y, e^x \cos y\rangle$, $\nabla f(P) = \langle e, 0\rangle$

① $\vec{u} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle$ $D_{\vec{u}} f(1, \pi/2) = \frac{-e}{\sqrt{2}}$

② $-\|\nabla f(P)\| = -e$

(c) $\nabla f(x,y) = \langle e^{-y}, -xe^{-y}\rangle$ $\nabla f(P) = \langle 1, -1\rangle$

① $\nabla f(P)\cdot\langle 0, 1\rangle = -1$

② 0

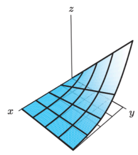
(d) $\nabla f(x,y) = \left\langle \frac{x}{\sqrt{x^2+2y}}, \frac{1}{\sqrt{x^2+2y}}\right\rangle$

$\nabla f(P) = \left\langle \frac{4}{\sqrt{16+20}}, \frac{1}{6}\right\rangle = \left\langle \frac{2}{3}, \frac{1}{6}\right\rangle$

① $\nabla f(P)\cdot\langle 1, 0\rangle = \frac{2}{3}$

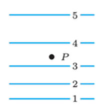
② 0

Problem 2. The figure below shows the graph of $f(x, y)$ on the domain $0 \le x \le 4$ and $0 \le y \le 4$. Use the graph to rank the following quantities from smallest to largest: $f_x(3,2), f_x(1,2), f_y(3,2), f_y(1,2), 0$.



$$f_x(1,2) < f_x(3,2) < 0 < f_y(3,2) < f_y(1,2)$$

Problem 3. The figure below shows a contour plot of $f(x, y)$. Use the contour plot to determine the sign (positive, negative, or zero) of $f_x(P), f_y(P), f_{xx}(P), f_{yy}(P), f_{xy}(P)$.



deriv.	sign
$f_x(P)$	0
$f_y(P)$	+
$f_{xx}(P)$	0
$f_{yy}(P)$	−
$f_{xy}(P)$	0

Problem 4. Consider the following functions. Find their critical points and use the Second Derivative Test to classify them.

a. $f(x, y) = y^2 + xy + 3y + 2x + 3$

b. $f(x, y) = x^2 + 2y^2 - x^2y$

ⓐ
$$\begin{cases} f_x = y + 2 = 0 \\ f_y = 2y + x + 3 = 0 \end{cases} \Rightarrow \begin{array}{l} y = -2, \\ x = -2y - 3 = 4 - 3 = 1 \end{array}$$

$f_{xx} = 0, \quad f_{yy} = 2, \quad f_{xy} = 1$

$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = -1$

critical point	D	f_{xx}	classification
(1, -2)	-1		saddle point

ⓑ
$$\begin{cases} f_x = 2x - 2xy = 0 & ① \\ f_y = 4y - x^2 = 0 & ② \end{cases}$$

② $\Rightarrow y = x^2/4$,

① $2x\left(1 - \frac{x^2}{4}\right) = 0$

$\Rightarrow 2x(4 - x^2) = 0$

$\Rightarrow 2x(2 - x)(2 + x) = 0$

$\Rightarrow x = -2, 0, 2$

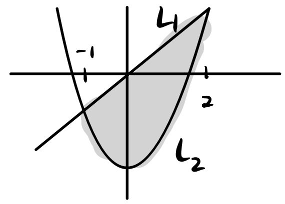
$f_{xx} = 2 - 2y, \quad f_{yy} = 4, \quad f_{xy} = -2x$

$D(x, y) = 4(2 - 2y) - (-2x)^2$

$= 4(2 - 2y) - 4x^2$

critical point	D	f_{xx}	classification
(-2, 1)	-16		saddle point
(0, 0)	8	2	local min
(2, 1)	-16		saddle point

Problem 5. Find the global minimum and maximum value, as well as the location of where they occur, of the function $f(x, y) = 3y - 2x$ constrained to the region bounded by $y = x^2 - 2$ and $y = x$.



$$\nabla f = \langle -2, 3 \rangle$$

no critical points

$$x^2 - 2 = x$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

$\underline{L_1}$ $\quad y = x, \quad -1 \leq x \leq 2$

$$g_1(x) = f(x, x) = 3x - 2x = x \quad \text{on} \quad [-1, 2]$$

$g_1'(x) = 1$, no critical points

$$g(-1) = -1$$
$$g(2) = 2$$

$\underline{L_2}$ $\quad y = x^2 - 2, \quad -1 \leq x \leq 2$

$$g_2(x) = f(x, x^2 - 2) = 3(x^2 - 2) - 2x$$
$$= 3x^2 - 2x - 6$$

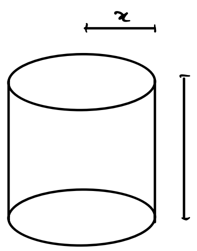
$g_2'(x) = 6x - 2$, $\quad x = \frac{1}{3}$ is critical point

$$g_2(-1) = -1$$
$$g_2\left(\tfrac{1}{3}\right) = 3\left(\tfrac{1}{3}\right)^2 - 2\left(\tfrac{1}{3}\right) - 6 = \frac{-19}{3}$$
$$g_2(2) = 2$$

global min value is $\frac{-19}{3}$, occurs at $\left(\frac{1}{3}, \frac{-17}{9}\right)$

global max value is 2, occurs at $(2, 2)$

Problem 6. A food manufacturing company is planning to make cans in the shape of circular cylinders with a volume of 10 cubic centimeters. In order to save on the cost of labels (on sides as well as tops and bottoms), they would like to minimize the surface area of the can. Use the method of Lagrange multipliers to find dimensions (radius x and height y) that fit these requirements.



minimize surface area

$$f(x,y) = 2\pi x^2 + 2\pi x y$$

with the constraint

$$g(x,y) = 10 \quad \text{where } g(x,y) = \pi x^2 y$$

$$\text{and } x, y > 0$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 10 \end{cases} \Rightarrow \begin{cases} 4\pi x + 2\pi y = \lambda(2\pi x y) \\ 2\pi x = \lambda(\pi x^2) \\ \pi x^2 y = 10 \end{cases}$$

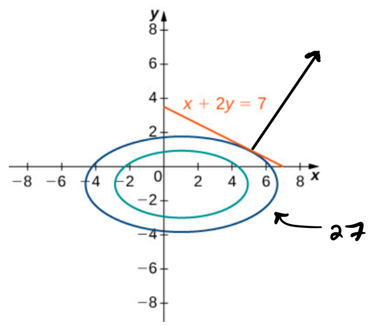
$$\Rightarrow \begin{cases} 2x + y = \lambda x y \\ 2 = \lambda x \Rightarrow x = \frac{2}{\lambda} \\ \pi x^2 y = 10 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{4}{\lambda} + y = 2y \Rightarrow y = \frac{4}{\lambda} \\ \pi \left(\frac{2}{\lambda}\right)^2 y = 10 \end{cases}$$

$$\Rightarrow \frac{16\pi}{\lambda^3} = 10 \Rightarrow \lambda = \left(\frac{8\pi}{5}\right)^{1/3}$$

$$x = 2\left(\frac{5}{8\pi}\right)^{1/3}, \quad y = 4\left(\frac{5}{8\pi}\right)^{1/3}$$

Problem 7. Let $f(x, y) = x^2 + 4y^2 - 2x + 8y$. Use the method of Lagrange Multipliers to determine the extreme values of f under the constraint that $x + 2y = 7$ and $x, y \geq 0$. In the figure below, label the z value of the level curve tangent to the line $x + 2y = 7$ and sketch in a vector in the direction of the gradient of f at the point where that level curve and line meet.



$$\begin{cases} \nabla f = \lambda \nabla g \\ x + 2y = 7 \end{cases} \Rightarrow \begin{cases} 2x - 2 = \lambda & (1) \\ 8y + 8 = 2\lambda & (2) \\ x + 2y = 7 & (3) \end{cases}$$

$(1), (2) \Rightarrow 2x - 2 = 4y + 4,$

$x = 2y + 3,$

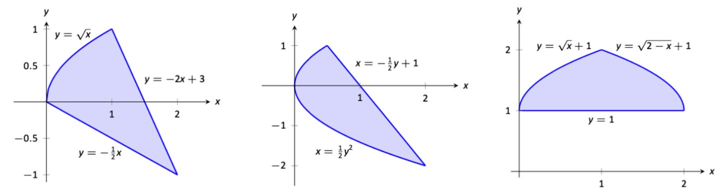
$(3) \Rightarrow 2y + 3 + 2y = 7, \quad y = 1, \quad x = 5$

$f(5,1) = 25 + 4 - 10 + 8 = 27$, min value

$f(7,0) = 49 - 14 = 35$

$f(0, \frac{7}{2}) = 4\left(\frac{7}{2}\right)^2 + 8\left(\frac{7}{2}\right) = 49 + 28 = 77$, max value

Problem 8. Consider each planar region R described below. Set up double integrals $\iint_R 1\,dA$ for the area of R in two ways: using $dA = dx\,dy$ and $dA = dy\,dx$.



(a) (b) (c)

(a)

$$\int_0^1 \int_{-\frac{1}{2}x}^{\sqrt{x}} dy\,dx + \int_1^2 \int_{-\frac{1}{2}x}^{-2x+3} dy\,dx$$

$$\int_{-1}^0 \int_{-2y}^{-\frac{1}{2}(y-3)} dx\,dy + \int_0^1 \int_{y^2}^{-\frac{1}{2}(y-3)} dx\,dy$$

(b)

$$\int_0^{1/2} \int_{-\sqrt{2x}}^{\sqrt{2x}} dy\,dx + \int_{1/2}^2 \int_{-\sqrt{2x}}^{-2(x-1)} dy\,dx$$

$$\int_{-2}^1 \int_{\frac{1}{2}y^2}^{-\frac{1}{2}y+1} dx\,dy$$

(c)

$$\int_0^1 \int_1^{\sqrt{x}+1} dy\,dx + \int_1^2 \int_1^{\sqrt{2-x}+1} dy\,dx$$

$$\int_1^2 \int_{(y-1)^2}^{2-(y-1)^2} dx\,dy$$

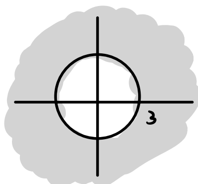
Problem 9. For each description in polar coordinates below, make a sketch of the given region.

a. $r \geq 3$

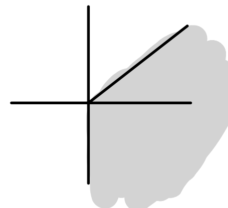
b. $-\pi/2 \leq \theta \leq \pi/4$

c. $1 \leq r \leq 2, 3\pi/4 \leq \theta \leq 7\pi/4$

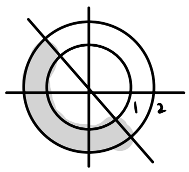
(a)



(b)



(c)



Problem 10. For each region in the xy-plane described below, express it using inequalities involving the polar variables r and θ.

a. The bottom half of the unit disk (centered at the origin)

b. The right half of the annulus with inner and outer radius 4 and 9 (centered at the origin)

c. The quarter of the unit disk in the third and fourth quadrants between the lines $y = \pm x$.

(a) $0 \leq r \leq 1$, $\pi \leq \theta \leq 2\pi$

(b) $4 \leq r \leq 9$ $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(c) $0 \leq r \leq 1$ $\frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4}$