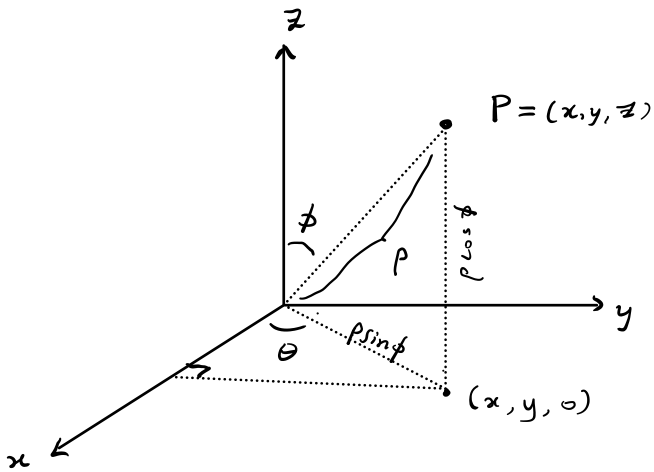
§13.7 Spherical Coordinates



ϕ measures angle with positive z-axis

θ measures polar angle in xy-plane

ρ measures dist. to $(0, 0, 0)$

A point $P = (x, y, z)$ in $\mathbb{R}^3$ can be represented by the quantities (ρ, θ, ϕ) in the diagram above, these are the spherical coordinates:

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad (\rho \geq 0)$$

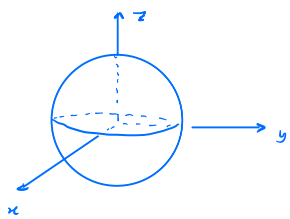
$$\tan\theta = \frac{y}{x} \qquad (0 \leq \theta \leq 2\pi)$$

$$\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}} \qquad (0 \leq \phi \leq \pi)$$

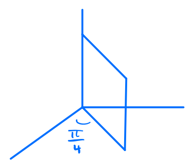
$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

Example Identify the surface described in spherical coordinates:

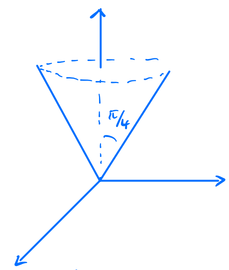
① $\rho = 1$, ② $\theta = \frac{\pi}{4}$, ③ $\phi = \frac{\pi}{4}$



unit sphere



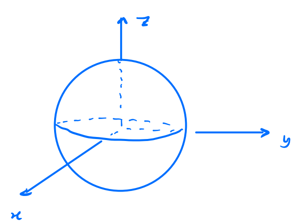
half plane



cone

Example Identify the solid described in spherical coordinates.

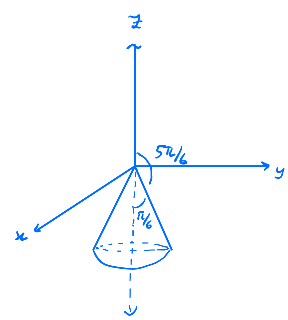
① $\rho \le 1$



unit ball

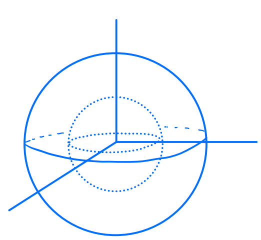
(the solid whose boundary is the unit sphere)

② $\rho \le 1$, $\frac{5\pi}{6} \le \phi \le \pi$



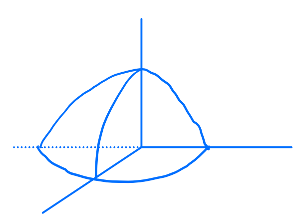
upside down solid ice cream cone

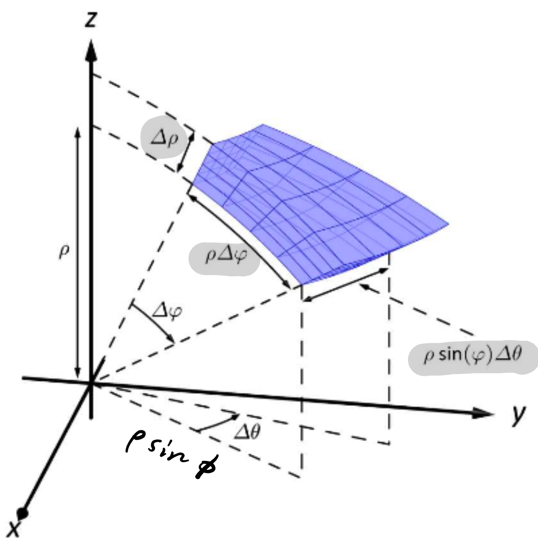
③ $1 \le \rho \le 2$


spherical shell

(solid region between two spheres)

④ $\rho \le 1$, $0 \le \phi \le \frac{\pi}{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$


top, front quarter of unit ball.



In spherical coordinates,

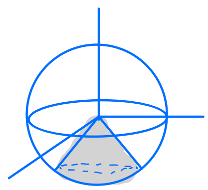
$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

$$\iiint f(x,y,z)\, dV$$

$$= \iiint f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

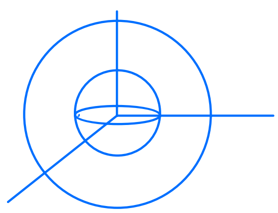
Example Set up triple integral for the volume of each solid described below.

① Below $z = -\sqrt{x^2+y^2}$ and above $z = -\sqrt{1-x^2-y^2}$



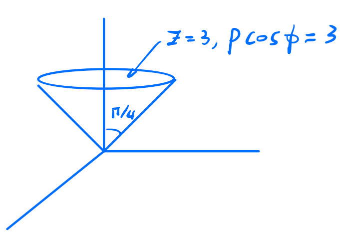
$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=3\pi/4}^{\phi=\pi}\int_{\rho=0}^{\rho=1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

② Between $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$ with $x \le 0, y \le 0$, and $z \le 0$.



$$\int_{\theta=\pi}^{\theta=3\pi/2}\int_{\phi=\pi/2}^{\phi=\pi}\int_{\rho=1}^{\rho=3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

③ Below $z = 3$ and above $z = \sqrt{x^2+y^2}$



$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=3\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Problem 1. For each point (ρ, θ, ϕ) given in spherical coordinates below, identify the sign of each component of its Cartesian coordinates (x, y, z). For example, if the point has positive x, negative y, and $z = 0$ your answer should be $(+, -, 0)$.

a. $(1, \pi/4, \pi/4)$

b. $(2, \pi, 3\pi/4)$

c. $(3, 5\pi/4, \pi/2)$

d. $(4, 7\pi/4, 5\pi/6)$

e. $(5, \pi/\cancel{2}\,4, \pi)$

ⓐ $(+,+,+)$ ⓒ $(-,-,0)$ ⓔ $(0,0,-)$

ⓑ $(-,0,-)$ ⓓ $(+,-,-)$

Problem 2. Describe the following regions using inequalities involving spherical variables ρ, θ, ϕ.

a. The quarter ball of radius 1, centered at the origin where $y \le 0$ and $z \le 0$.

b. The top half the solid region between spheres of radius 1 and 2 centered at the origin.

c. The plane $z = 1$.

d. The solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$.

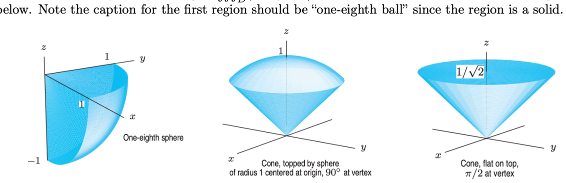
ⓐ $\pi \le \theta \le 2\pi, \quad \frac{\pi}{2} \le \phi \le \pi, \quad 0 \le \rho \le 1$

ⓑ $0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 1 \le \rho \le 2$

ⓒ $0 \le \theta \le 2\pi, \quad 0 \le \phi < \frac{\pi}{2}, \quad \rho = \sec\phi$

ⓓ $0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4, \quad 0 \le \rho \le \sec\phi$

Problem 3. Set up a triple integral $\iiint_D (x + y + z)\, dV$ in spherical coordinates for each region below. Note the caption for the first region should be "one-eighth ball" since the region is a solid.



ⓐ $$\int_0^{\pi/2}\int_{\pi/2}^{\pi}\int_0^1 (\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

ⓑ $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

ⓒ $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi/\sqrt{2}} (\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$