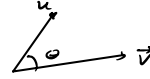


14.3 Line Integrals of Vector Fields

Warm up



Recall $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$, so

- $\vec{u} \cdot \vec{v} = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
- $\vec{u} \cdot \vec{v} > 0 \Rightarrow \cos \theta > 0 \Rightarrow 0 \leq \theta < \frac{\pi}{2}$
- $\vec{u} \cdot \vec{v} < 0 \Rightarrow \cos \theta < 0 \Rightarrow \frac{\pi}{2} < \theta \leq \pi$

Definition An oriented curve is a curve where there is a specified direction of travel (eg. a parametrized curve that goes counterclockwise).

A curve is simple if it does not cross itself.

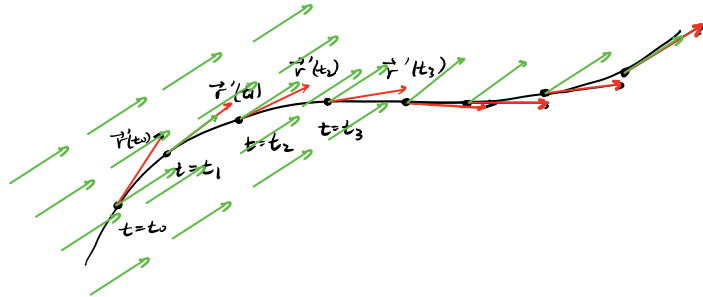
Given a simple, oriented curve C , parametrized by $\vec{r}(t)$, $a \leq t \leq b$, and a vector field \vec{F} .

the line integral of F over C is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

If C is closed (meaning it starts and ends in same place, ie. it's a loop), we write

$$\oint_C \vec{F} \cdot d\vec{r}.$$



- Divide C into small segments at times

$$a = t_0 < t_1 < t_2 < \dots < t_n = b$$

- Measure how much \vec{F} and $\vec{r}'(t)$ point in the same direction at each time t_i

$$\vec{F}(\vec{r}(t_i)) \cdot \vec{r}'(t_i)$$

- Sum the results and take a limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n \underbrace{\vec{F}(\vec{r}(t_i))}_{\text{force}} \cdot \underbrace{\vec{r}'(t_i) \Delta t}_{\text{distance}}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

= work done by F

Physical Interpretations of $\int_C \vec{F} \cdot d\vec{r}$

- ① If \vec{F} is a force and C is the trajectory of a mass under the influence of \vec{F} , the line integral is the work done by \vec{F} .
- ② If \vec{F} is the velocity field of a fluid flow and C is a closed curve, the line integral measures circulation (flow with or against C)

Properties of Line Integrals of Vector Fields

- ①
$$\int_C (a\vec{F} + b\vec{G}) \cdot d\vec{r} = a \int_C \vec{F} \cdot d\vec{r} + b \int_C \vec{G} \cdot d\vec{r}$$
- ② If C is piecewise defined and consists of curves C_1 and C_2 , then
- $$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$
- ③ If $-C$ is the reversed orientation of C , then
- $$\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$$

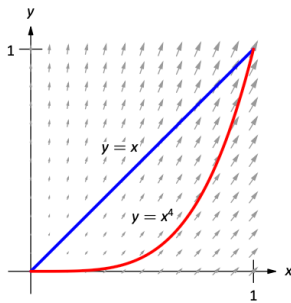
Example Let $\vec{F} = \langle x, x+y \rangle$, let C_1 and C_2

be oriented curves from $(0,0)$ to $(1,1)$, where

C_1 follows $y=x$ and C_2 follows $y=x^4$.

Let C be closed oriented that follows C_2 then $-C_1$.

Compute $\int_{C_1} \vec{F} \cdot d\vec{r}$, $\int_{C_2} \vec{F} \cdot d\vec{r}$, $\oint_C \vec{F} \cdot d\vec{r}$



$$\vec{r}_1(t) = \langle t, t \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_1'(t) = \langle 1, 1 \rangle$$

$$\vec{r}_2(t) = \langle t, t^4 \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_2'(t) = \langle 1, 4t^3 \rangle$$

$$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1'(t) = \langle t, 2t \rangle \cdot \langle 1, 1 \rangle = 3t$$

$$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2'(t) = \langle t, t+t^4 \rangle \cdot \langle 1, 4t^3 \rangle = t + 4t^4 + 4t^7$$

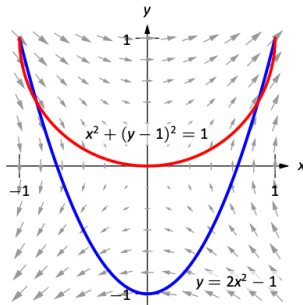
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 3t dt = \frac{3}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 (t + 4t^4 + 4t^7) dt = \frac{1}{2} + \frac{4}{5} + \frac{1}{2} = \frac{9}{2}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} = \frac{9}{2} - \frac{3}{2} = 3$$

($\oint_C \vec{F} \cdot d\vec{r} > 0$ means there's a net tendency to circulate with C)

Example Let $\vec{F}(x,y) = \langle y, x \rangle$, let C_1 and C_2 be oriented curves from $(-1,1)$ to $(1,1)$, where C_1 follows $y = 2x^2 - 1$ and the other follows the bottom half of the circle $x^2 + (y-1)^2 = 1$. Let C be the closed curve that follows C_1 then $-C_2$.



both oriented
from left
to right

$$\vec{r}_1(t) = \langle t, 2t^2 - 1 \rangle, \quad -1 \leq t \leq 1, \quad \vec{r}_1'(t) = \langle 1, 4t \rangle$$

$$\vec{r}_2(t) = \langle 0, 1 \rangle + \langle \cos t, \sin t \rangle$$

$$= \langle \cos t, 1 + \sin t \rangle, \quad -\pi \leq t \leq 0, \quad \vec{r}_2'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1'(t) = \langle 2t^2 - 1, t \rangle \cdot \langle 1, 4t \rangle = 6t^2 - 1$$

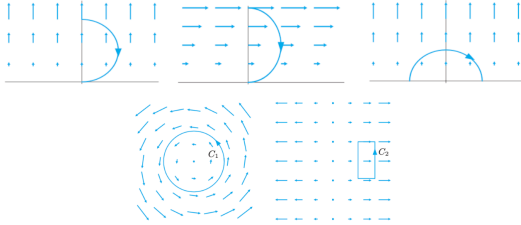
$$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2'(t) = \langle 1 + \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle = -\sin t - \sin^2 t + \cos^2 t$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-1}^1 (6t^2 - 1) dt = 4 - 2 = 2$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{-\pi}^0 (-\sin t - \sin^2 t + \cos^2 t) dt = 2$$

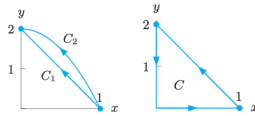
$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0. \quad \text{net circulation of } 0$$

Problem 1. Consider the vector fields \mathbf{F} and oriented curves C shown below. Determine the sign (positive, negative, or zero) of $\int_C \mathbf{F} \cdot d\mathbf{r}$.



-	+	0
+		0

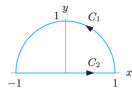
Problem 2. Let $\mathbf{F}(x, y) = (x, y)$. For each figure below, let C be the piecewise-defined closed curve that is oriented in a counter-clockwise fashion. Set up and compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Before doing your calculations, use CalcPlot3d to make a plot of \mathbf{F} and make a conjecture about the sign of the line integral. *Note: C_2 is part of the parabola $y = 2 - 2x^2$.*



$$\begin{aligned} \vec{r}_1(t) &= \langle t, 2 - 2t \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_1'(t) = \langle 1, -2 \rangle \\ \vec{r}_2(t) &= \langle t, 2 - 2t^2 \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_2'(t) = \langle 1, -4t \rangle \\ \oint_C \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \langle t, 2 - 2t \rangle \cdot \langle 1, -2 \rangle dt \\ &\quad - \int_0^1 \langle t, 2 - 2t^2 \rangle \cdot \langle 1, -4t \rangle dt \\ &= \int_0^1 (t - 4 + 4t) dt - \int_0^1 (t - 8t + 8t^3) dt \\ &= -1.5 + 1.5 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{r}_1(t) &= \langle 1, 0 \rangle + t \langle -1, 2 \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_1'(t) = \langle -1, 2 \rangle \\ \vec{r}_2(t) &= \langle 0, 2 \rangle + t \langle 0, -2 \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_2'(t) = \langle 0, -2 \rangle \\ \vec{r}_3(t) &= t \langle 0, 1 \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}_3'(t) = \langle 0, 1 \rangle \\ \oint_C \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{r}_1 + \int_{C_2} \vec{F} \cdot d\vec{r}_2 + \int_{C_3} \vec{F} \cdot d\vec{r}_3 \\ &= \int_0^1 \langle 1 - t, 2t \rangle \cdot \langle -1, 2 \rangle dt + \int_0^1 \langle 0, 2 - 2t \rangle \cdot \langle 0, -2 \rangle dt \\ &\quad + \int_0^1 \langle 0, t \rangle \cdot \langle 0, 1 \rangle dt \\ &= \int_0^1 (-1 + 5t) dt + \int_0^1 (-4 + 4t) dt + \int_0^1 t dt \\ &= 1.5 + (-2) + 0.5 \\ &= 0 \end{aligned}$$

Problem 3. Let $\mathbf{F}(x, y) = (-y, x)$. Let C be the piecewise-defined closed curve that is oriented in a counter-clockwise fashion as shown below. Set up and compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Before doing your calculation, use CalcPlot3d to make a plot of \mathbf{F} and make a conjecture about the sign of the line integral.



$$\vec{r}_1(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \pi, \quad \vec{r}_1'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{r}_2(t) = \langle t, 0 \rangle, \quad -1 \leq t \leq 1, \quad \vec{r}_2'(t) = \langle 1, 0 \rangle$$

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{r}_1 + \int_{C_2} \vec{F} \cdot d\vec{r}_2 \\ &= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt + \int_{-1}^1 \langle 0, t \rangle \cdot \langle 1, 0 \rangle dt \\ &= \pi \end{aligned}$$