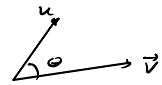
14.3 Line Integrals of Vector Fields

Warm up



Recall $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, so

- $\vec{u} \cdot \vec{v} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
- $\vec{u} \cdot \vec{v} > 0 \Rightarrow \cos\theta > 0 \Rightarrow 0 \leq \theta < \frac{\pi}{2}$
- $\vec{u} \cdot \vec{v} < 0 \Rightarrow \cos\theta < 0 \Rightarrow \frac{\pi}{2} < \theta \leq \pi$

__Definition__ An __oriented__ curve is a curve where there is a specified direction of travel (eg. a parametrized curve that goes counterclockwise).

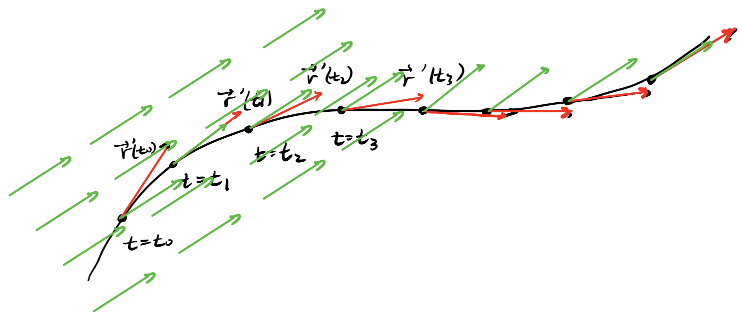
A curve is __simple__ if it does not cross itself.

Given a simple, oriented curve C, parametrized by $r(t)$, $a \leq t \leq b$, and a vector field $\vec{F}$, __the line integral of F over C__ is given by

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

If C is __closed__ (meaning it starts and ends in same place, ie. it's a loop), we write

$$\oint_C \vec{F} \cdot d\vec{r}.$$



- Divide C into small segments at times

$$a = t_0 < t_1 < t_2 < \dots < t_n = b$$

- Measure how much $\vec{F}$ and $\vec{r}'(t)$ point in the same direction at each time t_i

$$\vec{F}(\vec{r}(t_i)) \cdot \vec{r}'(t)$$

- Sum the results and take a limit as $n \to \infty$

$$\lim_{n\to\infty} \sum_{i=0}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}'(t_i)$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

Physical Interpretations of $\int_C \vec{F} \cdot d\vec{r}$

① If $\vec{F}$ is a force and C is the trajectory of a mass under the influence of $\vec{F}$, the line integral is the work done by $\vec{F}$.

② If $\vec{F}$ is the velocity field of a fluid flow and C is a closed curve, the line integral measures circulation (flow with or against C)

Properties of Line Integrals of Vector Fields

① $$\int_C (a\vec{F} + b\vec{G}) \cdot d\vec{r} = a\int_C \vec{F} \cdot d\vec{r} + b\int_C \vec{G} \cdot d\vec{r}$$

② If C is piecewise defined and consists of curves C_1 and C_2, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

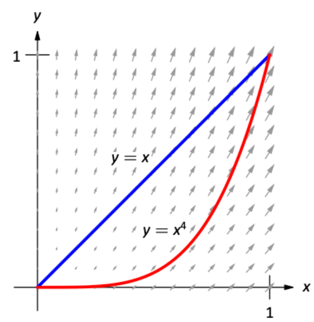
③ If $-C$ is the reversed orientation of C, then

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}.$$

Example Let $\vec{F} = \langle x, x+y \rangle$, let C_1 and C_2 be oriented curves from $(0,0)$ to $(1,1)$, where C_1 follows $y = x$ and C_2 follows $y = x^4$.

Let C be closed oriented that follows C_2 then $-C_1$.

Compute $\int_{C_1} \vec{F} \cdot d\vec{r}$, $\int_{C_2} \vec{F} \cdot d\vec{r}$, $\oint_C \vec{F} \cdot d\vec{r}$



$\vec{r}_1(t) = \langle t, t \rangle$, $0 \le t \le 1$, $\vec{r}_1{}'(t) = \langle 1, 1 \rangle$

$\vec{r}_2(t) = \langle t, t^4 \rangle$, $0 \le t \le 1$, $\vec{r}_2{}'(t) = \langle 1, 4t^3 \rangle$

$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = \langle t, 2t \rangle \cdot \langle 1, 1 \rangle = 3t$

$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) = \langle t, t + t^4 \rangle \cdot \langle 1, 4t^3 \rangle = t + 4t^4 + 4t^7$

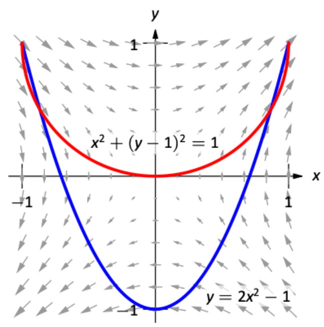
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 3t\,dt = \frac{3}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 (t + 4t^4 + 4t^7)\,dt = \frac{1}{2} + \frac{4}{5} + \frac{1}{2} = \frac{9}{2}.$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} = \frac{9}{2} - \frac{3}{2} = 3$$

($\oint_C \vec{F} \cdot d\vec{r} > 0$ means there's a <u>net</u> tendency to circulate with C)

Example Let $\vec{F}(x,y) = \langle y, x \rangle$, let C_1 and C_2 be oriented curves from $(-1,1)$ to $(1,1)$, where C_1 follows $y = 2x^2 - 1$ and the other follows the bottom half of the circle $x^2 + (y-1)^2 = 1$. Let C be the closed curve that follows C_1 then $-C_2$.



both oriented from left to right

$$\vec{r}_1(t) = \langle t, 2t^2 - 1 \rangle, \quad -1 \le t \le 1, \quad \vec{r}_1{}'(t) = \langle 1, 4t \rangle$$

$$r_2(t) = \langle 0, 1 \rangle + \langle \cos t, \sin t \rangle$$
$$= \langle \cos t, 1 + \sin t \rangle, \quad -\pi \le t \le 0, \quad \vec{r}_2{}'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = \langle 2t^2 - 1, t \rangle \cdot \langle 1, 4t \rangle = 6t^2 - 1$$

$$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) = \langle 1 + \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle = -\sin t - \sin^2 t + \cos^2 t$$

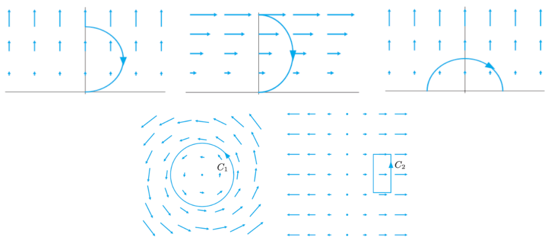
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} (6t^2 - 1)\, dt = 4 - 2 = 2$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{-\pi}^{0} (-\sin t - \sin^2 t + \cos^2 t)\, dt = 2$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0.$$

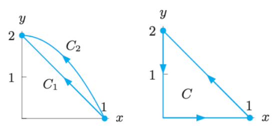
net circulation of 0

Problem 1. Consider the vector fields **F** and oriented curves C shown below. Determine the sign (positive, negative, or zero) of $\int_C \mathbf{F} \cdot d\mathbf{r}$.



$-$	$+$	0
$+$	0	

Problem 2. Let $\mathbf{F}(x, y) = \langle x, y \rangle$. For each figure below, let C be the piecewise-defined closed curve that is oriented in a counter-clockwise fashion. Set up and compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Before doing your calculations, use CalcPlot3d to make a plot of **F** and make a conjecture about the sign of the line integral. *Note: C_2 is part of the parabola $y = 2 - 2x^2$.*



$\vec{r}_1(t) = \langle t, 2-2t \rangle,\ 0 \le t \le 1,\ \vec{r}_1{}'(t) = \langle 1, -2 \rangle$

$\vec{r}_2(t) = \langle t, 2-2t^2 \rangle,\ 0 \le t \le 1,\ \vec{r}_2{}'(t) = \langle 1, -4t \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \langle t, 2-2t \rangle \cdot \langle 1, -2 \rangle dt - \int_0^1 \langle t, 2-2t^2 \rangle \cdot \langle 1, -4t \rangle dt$$

$$= \int_0^1 (t - 4 + 4t) dt - \int_0^1 (t - 8t + 8t^3) dt$$

$$= -1.5 + 1.5$$

$$= 0$$

$\vec{r}_1(t) = \langle 1, 0 \rangle + t \langle -1, 2 \rangle,\ 0 \le t \le 1,\ \vec{r}_1{}'(t) = \langle -1, 2 \rangle$

$\vec{r}_2(t) = \langle 0, 2 \rangle + t \langle 0, -2 \rangle,\ 0 \le t \le 1,\ \vec{r}_2{}'(t) = \langle 0, -2 \rangle$

$\vec{r}_3(t) = t \langle 0, 1 \rangle,\ 0 \le t \le 1,\ \vec{r}_3{}'(t) = \langle 0, 1 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}_1 + \int_{C_2} \vec{F} \cdot d\vec{r}_2 + \int_{C_3} \vec{F} \cdot d\vec{r}_3$$

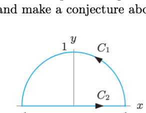
$$= \int_0^1 \langle 1-t, 2t \rangle \cdot \langle -1, 2 \rangle dt + \int_0^1 \langle 0, 2-2t \rangle \cdot \langle 0, -2 \rangle dt + \int_0^1 \langle 0, t \rangle \cdot \langle 0, 1 \rangle dt$$

$$= \int_0^1 (-1 + 5t) dt + \int_0^1 (-4 + 4t) dt + \int_0^1 t\, dt$$

$$= 1.5 + (-2) + 0.5$$

$$= 0$$

Problem 3. Let $\mathbf{F}(x, y) = \langle -y, x \rangle$. Let C be the piecewise-defined closed curve that is oriented in a counter-clockwise fashion as shown below. Set up and compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Before doing your calculation, use CalcPlot3d to make a plot of **F** and make a conjecture about the sign of the line integral.



$\vec{r}_1(t) = \langle \cos t, \sin t \rangle,\ 0 \le t \le \pi,\ \vec{r}_1{}'(t) = \langle -\sin t, \cos t \rangle$

$\vec{r}_2(t) = \langle t, 0 \rangle,\ -1 \le t \le 1,\ \vec{r}_2{}'(t) = \langle 1, 0 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}_1 + \int_{C_2} \vec{F} \cdot d\vec{r}_2$$

$$= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt + \int_{-1}^1 \langle 0, t \rangle \cdot \langle 1, 0 \rangle dt$$

$$= \pi$$