10.3 Dot Product

Def Let $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$, $\vec{w} = \langle w_1, \dots, w_n \rangle \in \mathbb{R}^n$

Their dot product is the scalar given by

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n .$$

We will soon see that the dot product helps us think about the angle between two vectors.

Example Compute $\vec{v} \cdot \vec{w}$ when $\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{w} = \langle 4, 5, 6 \rangle$.

$$\vec{v} \cdot \vec{w} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Notice the result is a scalar (ie. number) not a vector.

Algebraic properties

① $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutative property)

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)

③ $c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w})$

④ $\vec{0} \cdot \vec{v} = 0$ note $\vec{0} = \langle 0, 0, \dots, 0 \rangle$

⑤ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

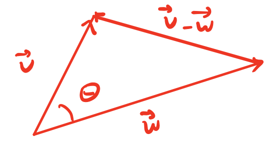
since $\vec{v} \cdot \vec{v} = v_1^2 + \cdots + v_n^2$

$$= \left(\sqrt{v_1^2 + \cdots + v_n^2} \right)^2$$

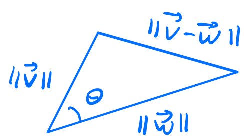
$$= \|\vec{v}\|^2$$

Formula Let θ be the angle in $[0,\pi]$ between the vectors $\vec{v}$ and $\vec{w}$. Then

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta.$$



Proof



The law of cosines tells us

$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta.$$

Moreover,
$$\begin{aligned}\|\vec{v}-\vec{w}\|^2 &= (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) \\ &= \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\,\vec{v}\cdot\vec{w} \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\,\vec{v}\cdot\vec{w}\end{aligned}$$

Combining these equations yields the formula.

Example Compute the angle between $\vec{v} = \langle 1,0,2\rangle$ and $\vec{w} = \langle -1,1,4\rangle$.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right) \\ &= \cos^{-1}\left(\frac{-1+0+8}{\sqrt{5}\cdot\sqrt{18}}\right) \\ &= \cos^{-1}\left(\frac{7}{\sqrt{90}}\right)\end{aligned}$$

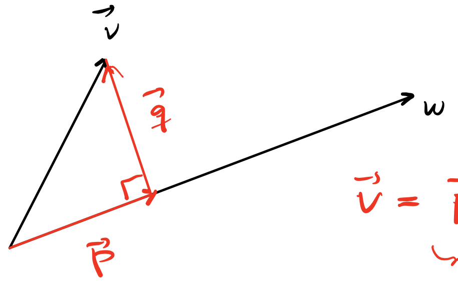
Important idea If $\vec{v}\cdot\vec{w} = 0$, then $\theta = \frac{\pi}{2}$.

We say $\vec{v}$ and $\vec{w}$ are **orthogonal** (ie perpendicular)

Also, $\vec{v}\cdot\vec{w} > 0 \iff 0 \le \theta < \frac{\pi}{2}$ and $\vec{v}\cdot\vec{w} < 0 \iff \frac{\pi}{2} < \theta \le \pi$

Orthogonal Projection

Given vector $\vec{v}$ we sometimes want to "decompose" it in terms of another vector $\vec{w}$



perpendicular to $\vec{w}$

$$\vec{v} = \underbrace{\vec{p}}_{\text{parallel to } \vec{w}} + \overbrace{\vec{q}}^{\text{perpendicular to } \vec{w}}$$

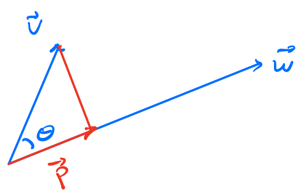
parallel to $\vec{w}$

The vector $\vec{p}$ in the sketch above is called the <u>orthogonal projection</u> of $\vec{v}$ <u>onto</u> $\vec{w}$.

<u>Notation</u> $\quad \text{proj}_{\vec{w}} \vec{v}$

<u>Formula</u> $\quad \text{proj}_{\vec{w}} \vec{v} = \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$

<u>Proof</u>



By trigonometry $\cos\theta = \dfrac{\|\vec{p}\|}{\|\vec{v}\|}$, so

$$\|\vec{p}\| = \|\vec{v}\| \cos\theta$$

$$= \|\vec{v}\| \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \quad \text{by dot product angle formula}$$

$$= \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}.$$

Note $p = c\vec{w}$ for some c and so

$$\|c\vec{w}\| = \|p\| = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$$

which implies $|c| = \dfrac{\vec{v}\cdot\vec{w}}{\|w\|^2} = \dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$ and

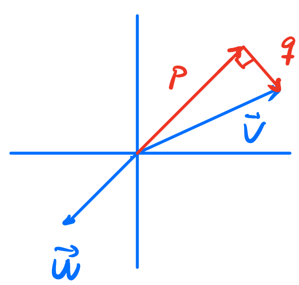
so $p = \left(\dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$ (details about abs. value omitted)

<u>Example</u> Let $\vec{v} = \langle 2, 1\rangle$ and $\vec{w} = \langle -1, -1\rangle$

Find vectors $\vec{p}$ and $\vec{q}$ so that $\vec{v} = \vec{p} + \vec{q}$

and $\vec{p}$ and $\vec{q}$ are parallel and perpendicular

to $\vec{w}$.



$$\vec{p} = \mathrm{proj}_{\vec{w}}\,\vec{v} = \left(\frac{-2-1}{2}\right)\vec{w} = -\frac{3}{2}\langle -1,-1\rangle = \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle$$

$$\vec{q} = \vec{v} - \vec{p} = \langle 2, 1\rangle - \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

Problem 1. Find the angle between the following vectors. You can leave your answer in terms of $\cos^{-1}$ but make note if any vectors are orthogonal (in which case the angle is $\pi/2$).

a. $\langle 3, 1\rangle$ and $\langle -4, 3\rangle$

b. $\langle 1, 2\rangle$ and $\langle 2, -1\rangle$

c. $\langle 1, -8, 2\rangle$ and $\langle 0, 1, 4\rangle$

(a) $\cos^{-1}\left(\frac{-12+3}{\sqrt{10}\sqrt{25}}\right) = \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right)$

(b) dot product : $2 + (-2) = 0$, so orthogonal

(c) dot product $= 0 - 8 + 8 = 0$, so orthogonal

Problem 2. Find the orthgonal projection of $\langle 1, 1, 2\rangle$ onto $\langle 0, 1, 1\rangle$.

$$\left(\frac{0+1+2}{0^2+1^2+1^2}\right)\langle 0, 1, 1\rangle = \left\langle 0, \frac{3}{2}, \frac{3}{2}\right\rangle$$

Problem 3. Consider the vectors $\mathbf{v} = \langle 1, 5\rangle$ and $\mathbf{w} = \langle 4, 1\rangle$.

a. Make a sketch $\mathbf{v}$ and $\mathbf{w}$ on the same set of axes.

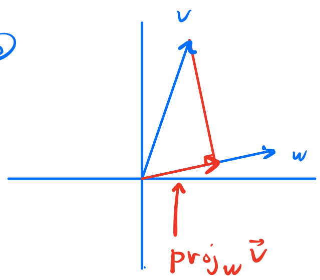
b. Make a sketch of the $\text{proj}_{\mathbf{w}}\,\mathbf{v}$.

c. Make a sketch of the $\text{proj}_{\mathbf{v}}\,\mathbf{w}$.

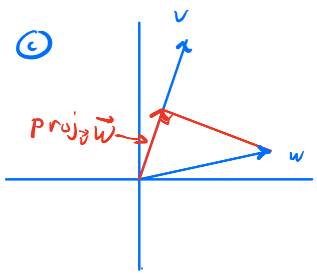
d. Find vectors $\mathbf{p}$ and $\mathbf{q}$ so that $\mathbf{v} = \mathbf{p} + \mathbf{q}$ where $\mathbf{p}$ is parallel to $\mathbf{w}$ and $\mathbf{q}$ is perpendicular to $\mathbf{w}$.

e. Find vectors $\mathbf{r}$ and $\mathbf{s}$ so that $\mathbf{w} = \mathbf{r} + \mathbf{s}$ where $\mathbf{r}$ is parallel to $\mathbf{v}$ and $\mathbf{s}$ is perpendicular to $\mathbf{v}$.

(b)



(c)



(d) $$\vec{p} = \text{proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

$$= \left(\frac{4+5}{17}\right)\langle 4,1\rangle = \left\langle \frac{36}{17}, \frac{9}{17}\right\rangle$$

$$q = \vec{v} - \vec{p} = \langle 1,5\rangle - \left\langle \frac{36}{17}, \frac{9}{17}\right\rangle$$

$$= \left\langle \frac{-19}{17}, \frac{76}{17}\right\rangle$$

(e) $$\vec{r} = \text{proj}_{\vec{v}} \vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$= \left(\frac{9}{26}\right)\langle 1,5\rangle = \left\langle \frac{9}{26}, \frac{45}{26}\right\rangle$$

$$\vec{s} = \vec{w} - \vec{r} = \langle 4,1\rangle - \left\langle \frac{9}{26}, \frac{45}{26}\right\rangle$$

$$= \left\langle \frac{95}{26}, \frac{-19}{26}\right\rangle$$