

### 14.3 More Line Integrals

The fundamental theorem of calculus from calculus I, II says  $\int_a^b f'(x) dx = f(b) - f(a)$

#### Theorem (Fundamental Theorem of Line Integrals)

If  $C$  is a piecewise smooth curve connecting  $A$  and  $B$  and  $\nabla f$  is continuous along  $C$  then

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

Informal derivation Let  $C$  be parametrized by  $\vec{r}(t) = \langle x(t), y(t) \rangle$

f.  $a \leq t \leq b$ . Then

$$\begin{aligned} \int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_a^b \frac{d}{dt} (f(\vec{r}(t))) dt \\ &= f(\vec{r}(t)) \Big|_a^b \\ &= f(B) - f(A) \end{aligned}$$

$$\begin{aligned} \text{Note } \frac{d}{dt} (f(\vec{r}(t))) &= f_x(\vec{r}(t)) x'(t) + f_y(\vec{r}(t)) y'(t) \\ &= \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \end{aligned}$$

is called the multivariable chain rule.

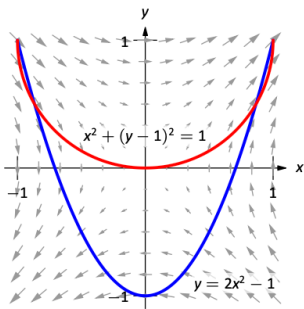
Remark Notice  $\int_C \nabla f \cdot d\vec{r}$  only depends on the endpoints.

if  $\vec{F}$  represents force, path-independence means work done by  $\vec{F}$  only depends on start and end points (like when  $\vec{F}$  is force due to gravity)

Def  $\vec{F}$  is called path-independent if  $\int_C \vec{F} \cdot d\vec{r}$  is the same value for all piecewise smooth curves  $C$  from any  $A$  to  $B$  which lie in the domain of  $\vec{F}$ .

$\vec{F}$  is called conservative if  $\vec{F} = \nabla f$  for some function  $f$  (called a potential function of  $\vec{F}$ ).  
 "conservative" comes from "conservation of energy"

Theorem A continuous vector field  $\vec{F}$  is path-independent if and only if it is conservative.



Example Consider the vector field  $\vec{F} = \langle y, x \rangle$ .

Show  $\vec{F} = \nabla f$  for  $f(x, y) = xy$  and

compute  $\int_{C_1} \vec{F} \cdot d\vec{r}$  and  $\int_{C_2} \vec{F} \cdot d\vec{r}$

where  $C_1$  and  $C_2$  are the oriented curves

from  $(-1, 1)$  to  $(1, 1)$  in red and blue on the right respectively.

Note  $\vec{F} = \nabla f$  since  $f_x = y, f_y = x$ . Therefore

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = f(1, 1) - f(-1, 1) = 1 - (-1) = 2.$$

Remark When  $\vec{F}$  is conservative,  $\oint_C \vec{F} \cdot d\vec{r} = 0$  for any simple closed curve.

Example Let  $\vec{F}(x,y) = \langle 3x^2y + 2x, x^3 + 1 \rangle$ .

Show that  $\vec{F}$  has a potential function by finding one.

Need to find  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  so that  $\nabla f = \vec{F}$ :

$$\begin{cases} f_x = 3x^2y + 2x \\ f_y = x^3 + 1 \end{cases}$$

$$\begin{aligned} \text{Note } f(x,y) &= \int f_y dy \\ &= \int (x^3 + 1) dy \\ &= y(x^3 + 1) + C_1(x) \\ &= x^3y + y + C_1(x) \end{aligned}$$

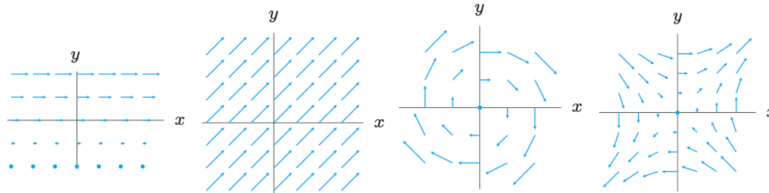
$$\begin{aligned} \text{and } f(x,y) &= \int f_x dx \\ &= \int (3x^2y + 2x) dx \\ &= x^3y + x^2 + C_2(y) \end{aligned}$$

To make this work we can choose  $C_1(x) = x^2$

and  $C_2(y) = y$ , so that

$$f(x,y) = x^3y + x^2 + y$$

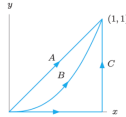
**Problem 1.** Let  $C$  be the closed curve consisting of a square centered at the origin with vertices at  $(\pm 1, \pm 1)$ , oriented counter-clockwise. Determine the sign (positive, negative, or zero) of  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  for the vector fields  $\mathbf{F}$  shown below. Indicate whether the vector field is path-independent.



sign	-	0	-	0
path-independent	no	yes	no	yes

**Problem 2.** For each of the following vector fields  $\mathbf{F}$ , find a potential function  $f$  of  $\mathbf{F}$  and use the Fundamental Theorem of Line Integrals to compute  $\int_A \mathbf{F} \cdot d\mathbf{r}$ ,  $\int_B \mathbf{F} \cdot d\mathbf{r}$ , and  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $A$ ,  $B$ , and  $C$  are the curves shown below.

- $\mathbf{F}(x, y) = (x, y)$
- $\mathbf{F}(x, y) = (y \cos x, \sin x + y)$
- $\mathbf{F}(x, y) = (2xy, x^2)$
- $\mathbf{F}(x, y) = (2xy, x^2 + 8y^2)$



$$\textcircled{a} \quad f_x = x \Rightarrow f(x, y) = \int x dx = \frac{1}{2}x^2 + C_1(y),$$

$$f_y = y \Rightarrow f(x, y) = \int y dy = \frac{1}{2}y^2 + C_2(x)$$

$$\Rightarrow f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 \quad (+ \text{any constant})$$

$$\int_A \vec{F} \cdot d\vec{r} = \int_B \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = 1$$

$$\textcircled{b} \quad f_x = y \cos x \Rightarrow f(x, y) = \int y \cos x dx = y \sin x + C_1(y)$$

$$f_y = \sin x + y \Rightarrow f(x, y) = \int (\sin x + y) dy = y \sin x + \frac{1}{2}y^2 + C_2(x)$$

$$\Rightarrow C_1(y) = \frac{1}{2}y^2, \quad C_2(x) = 0 \quad (\text{or any constant})$$

$$\Rightarrow f(x, y) = y \sin x + \frac{1}{2}y^2$$

$$\int_A \vec{F} \cdot d\vec{r} = \int_B \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = \sin(1) + \frac{1}{2}$$

$$\textcircled{c} \quad f_x = 2xy \Rightarrow f(x, y) = \int 2xy dx = x^2 y + C_1(y)$$

$$f_y = x^2 \Rightarrow f(x, y) = \int x^2 dy = x^2 y + C_2(x)$$

$$\Rightarrow C_1(y) = C_2(x) = 0 \quad (\text{or any constant})$$

$$\Rightarrow f(x, y) = x^2 y$$

$$\int_A \vec{F} \cdot d\vec{r} = \int_B \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = 1$$

$$\textcircled{d} \quad f_x = 2xy \Rightarrow f(x,y) = x^2y + C(y)$$

$$x^2 + 8y^3 = f_y = x^2 + C'(y)$$

$$\Rightarrow C'(y) = 8y^3$$

$$\Rightarrow C(y) = \int 8y^3 dy = 2y^4 \quad (+ \text{ any constant})$$

$$f(x,y) = x^2y + 2y^4$$

$$\int_A \vec{F} \cdot d\vec{r} = \int_B \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = 3$$

**Problem 3.** Try finding a potential function for  $\mathbf{F}(x,y) = \langle -y, x \rangle$ . What goes wrong?

$$\vec{F}(x,y) = \langle -y, x \rangle$$

$$\Rightarrow f_x = -y \Rightarrow f(x,y) = \int -y dx = -xy + C_1(y)$$

$$f_y = x \Rightarrow f(x,y) = \int x dy = xy + C_2(x)$$

But there's no possible choice of  $C_1(y), C_2(x)$

$$\text{since } f_{xy} = -x + C_1'(y) \text{ and } f_{yx} = x + C_2'(x)$$

which means  $C_1'(y) = 2x$ , but this is impossible since  $C_1$

can only be a function of  $y$

**Problem 4.** Compute  $\int_A \mathbf{F} \cdot d\mathbf{r}$  and  $\int_C \mathbf{F} \cdot d\mathbf{r}$  for the vector field  $\mathbf{F}(x,y) = \langle -y, x \rangle$  using the curves  $A$  and  $C$  from Problem 2. Is  $\mathbf{F}$  path-independent?

$$\textcircled{A} \quad \vec{r}(t) = \langle t, t \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}'(t) = \langle 1, 1 \rangle$$

$$\textcircled{C} \quad \vec{r}(t) = \begin{cases} \langle t, 0 \rangle & 0 \leq t \leq 1 \\ \langle 1, t-1 \rangle & 1 \leq t \leq 2 \end{cases} \quad \vec{r}'(t) = \begin{cases} \langle 1, 0 \rangle & 0 \leq t \leq 1 \\ \langle 0, 1 \rangle & 1 \leq t \leq 2 \end{cases}$$

$$\int_A \vec{F} \cdot d\vec{r} = \int_0^1 \langle -t, t \rangle \cdot \langle 1, 1 \rangle dt = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle 0, t \rangle \cdot \langle 1, 0 \rangle dt + \int_1^2 \langle 1-t, 1 \rangle \cdot \langle 0, 1 \rangle dt = 1$$