# 14.4 Divergence Theorem in 2D

**Definition** Let $\vec{F} = \langle M, N \rangle$ be a smooth vector field. Then the divergence of $\vec{F}$ is $\operatorname{div} \vec{F} = M_x + N_y$.

**Theorem** (Divergence Theorem) Let $C$ be a piecewise smooth, positively oriented, closed curve that encloses a region $R \subseteq \mathbb{R}^2$. Let $\vec{F}$ be a smooth vector field whose domain includes $R$ and $C$. Then

$$\oint_C F \cdot \vec{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA .$$

The macroscopic flux $\oint_C \vec{F} \cdot \vec{n}\, ds$ across $C$ can be computed by summing microscopic fluxes throughout region $R$.

# Proof of the Divergence Theorem

Let $\vec{F} = \langle M, N \rangle$ and $\vec{G} = \langle -N, M \rangle$ and

let $C$ be parametrized by $\vec{r}(t) = \langle f(t), g(t) \rangle$

where $a \le t \le b$ so that $\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle$

and $\vec{n} = \langle g'(t), -f'(t) \rangle$. Notice

$$\text{curl}(\vec{G}) = M_x - (-N)_y = \text{div}(\vec{F})$$

Therefore

$$\iint_R \text{div}(F)\,dA = \iint_R \text{curl}(\vec{G})\,dA$$

$$= \oint_C \vec{G} \cdot d\vec{r} \quad \text{by Green's Theorem}$$

$$= \int_a^b \langle -N, M \rangle \cdot \langle f', g' \rangle \, dt$$

$$= \int_a^b (-Nf' + Mg')\,dt$$

$$= \int_a^b \langle M, N \rangle \cdot \langle g', -f' \rangle \, dt$$

$$= \oint_C \vec{F} \cdot \vec{n}\, ds$$

Example Let $\vec{F}(x,y) = \langle -x, 2y-x \rangle$ and

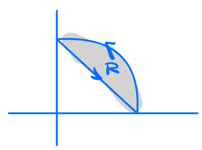
$C_1$ the quarter unit circle from $(1,0)$ to $(0,1)$,

$C_2$ the line segment from $(0,1)$ to $(1,0)$,

$C$ the closed, positively oriented curve $C_1 + C_2$,

$R$ the region enclosed by $C$. Verify the Divergence Theorem by computing both

$$\oint_C \vec{F} \cdot \vec{n}\, ds \quad \text{and} \quad \iint_R \operatorname{div} \vec{F}\, dA.$$




$\vec{r}_1(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \frac{\pi}{2}$

$\vec{r}_1{}'(t) = \langle -\sin t, \cos t \rangle$

$\vec{r}_2(t) = \langle t, 1-t \rangle, \quad 0 \le t \le 1$

$\vec{r}_2{}'(t) = \langle 1, -1 \rangle$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_{C_1} \vec{F} \cdot \vec{n}\, ds + \int_{C_2} \vec{F} \cdot \vec{n}\, ds$$

$$= \int_0^{\frac{\pi}{2}} \langle -\cos t, 2\sin t - \cos t \rangle \cdot \langle \cos t, \sin t \rangle\, dt$$

$$+ \int_0^1 \langle -t, 2(1-t) - t \rangle \cdot \langle -1, -1 \rangle\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left( -\cos^2 t + 2\sin^2 t - \sin t \cos t \right) dt$$

$$+ \int_0^1 \left( t + t - 2(1-t) \right) dt$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

$\operatorname{div} \vec{F} = -1 + 2 = 1$

$$\iint_R \operatorname{div} \vec{F} \cdot dA = \iint_R 1\, dA = \text{area}(R) = \frac{\pi}{4} - \frac{1}{2}$$

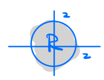
**Problem 1.** Let $\mathbf{F} = \langle x - y, x + y \rangle$, let $C$ be the circle of radius 2 centered at the origin, and let $R$ be the region enclosed by $C$. Verify the Divergence Theorem for this example. That is, compute $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ using a parametrization and compute $\iint_R \operatorname{div} F\, dA$.

$\vec{r}(t) = 2\langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, $\vec{r}\,'(t) = 2\langle -\sin t, \cos t\rangle$

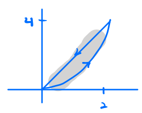
$$\vec{F}(\vec{r}(t)) \cdot \vec{n}(t)\, \|\vec{r}\,'(t)\| = 2\langle \cos t - \sin t, \cos t + \sin t\rangle \cdot 2\langle \cos t, \sin t\rangle$$
$$= 4(\cos^2 t - \sin t\cos t + \sin t\cos t + \sin^2 t)$$
$$= 4$$

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \int_0^{2\pi} 4\, dt = 8\pi$$

$\operatorname{div} F = 1 + 1 = 2$ $\qquad \iint_R \operatorname{div} F\, dA = 2\,\text{area}(R) = 2\pi(2)^2 = 8\pi$



**Problem 2.** Try Problem 1 using $C$ as the closed, positively oriented curve comprising the parabola $y = x^2$ for $0 \le x \le 2$ and the line segment from $(2, 4)$ to $(0, 0)$.



$\vec{r}_1(t) = \langle t, t^2\rangle$, $0 \le t \le 2$, $\vec{r}\,'(t) = \langle 1, 2t\rangle$

$\vec{r}_2(t) = \langle t, 2t\rangle$, $0 \le t \le 2$, $\vec{r}\,'(t) = \langle 1, 2\rangle$

$$\vec{F}(\vec{r}_1(t)) \cdot \vec{n}_1(t)\, \|\vec{r}_1\,'(t)\| = \langle t - t^2, t + t^2\rangle \cdot \langle 2t, -1\rangle = 2t^2 - 2t^3 - t - t^2$$
$$= -2t^3 + t^2 - t$$

$$\vec{F}(\vec{r}_2(t)) \cdot \vec{n}_2(t)\, \|\vec{r}_2\,'(t)\| = \langle t - 2t, t + 2t\rangle \cdot \langle 2, -1\rangle = -2t - 3t = -5t$$

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \int_0^2 (-2t^3 + t^2 + 4t)\, dt = -\tfrac{1}{2}t^4 + \tfrac{1}{3}t^3 + 2t^2 \Big|_0^2$$
$$= -8 + \frac{8}{3} + 8 = \frac{8}{3}$$

$$\iint_R \operatorname{div} F\, dA = \int_0^2 \int_{x^2}^{2x} 2\, dy\, dx$$
$$= 2\int_0^2 (2x - x^2)\, dx = 2\left(x^2 - \tfrac{1}{3}x^3 \Big|_0^2\right)$$
$$= 2\left(4 - \frac{8}{3}\right) = 2\left(\frac{12}{3} - \frac{8}{3}\right) = \frac{8}{3}$$

**Problem 3.** Let $\mathbf{F}$ be a vector field whose domain is all of $\mathbb{R}^2$ with the property that $\operatorname{div} \mathbf{F} = 0$ and let $C_1$ and $C_2$ be two non-intersecting curves each oriented so that they start from $(0, 0)$ and go to $(1, 1)$. Suppose $\int_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = 5$. Find the value of $\int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds$ and explain your reasoning.

Note if $R$ is the region enclosed by $C_1$ and $C_2$, $\iint_R \operatorname{div} F\, dA = 0$ since $\operatorname{div} F = 0$.

The divergence theorem tells us $0 = \iint_R \operatorname{div}\vec{F}\, dA = \oint_C \vec{F}\cdot\vec{n}\, ds = \int_{C_1} \vec{F}\cdot\vec{n}\, ds - \int_{C_2} \vec{F}\cdot\vec{n}\, ds$

$$= 5 - \int_{C_2} \vec{F}\cdot\vec{n}\, ds$$

Therefore $\int_{C_2} \vec{F}\cdot\vec{n}\, ds = 5$.