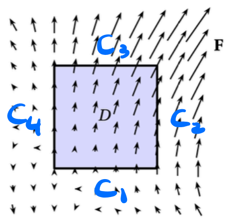
Problem 1. Let $\mathbf{F}(x,y) = \langle xy, x+y \rangle$, let C be the positive oriented square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and let D be the region enclosed by C. See the image below.

a. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by computing separate line integrals along the 4 sides of the square. To save time split the work up with a partner or two.

b. Compute $\iint_D \operatorname{curl} \mathbf{F}\, dA$ and compare with your previous answer. Do they match? Why?



(a)

$\vec{r}_1(t) = \langle t, 0 \rangle, \quad 0 \le t \le 1$

$\vec{r}_2(t) = \langle 1, t \rangle, \quad 0 \le t \le 1$

$\vec{r}_3(t) = \langle t, 1 \rangle, \quad 0 \le t \le 1$

$\vec{r}_4(t) = \langle 0, t \rangle, \quad 0 \le t \le 1$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_3} \vec{F} \cdot d\vec{r} - \int_{C_4} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \langle 0, t \rangle \cdot \langle 1, 0 \rangle\, dt + \int_0^1 \langle t, 1+t \rangle \cdot \langle 0, 1 \rangle\, dt$$

$$- \int_0^1 \langle t, 1+t \rangle \cdot \langle 1, 0 \rangle\, dt - \int_0^1 \langle 0, t \rangle \cdot \langle 0, 1 \rangle\, dt$$

$$= 0 + \int_0^1 (1+t)\, dt - \int_0^1 t\, dt - \int_0^1 t\, dt$$

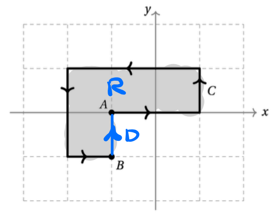
$$= 1 + \frac{1}{2} - 1 = \frac{1}{2}$$

(b) $\operatorname{curl} \vec{F} = 1 - x.$ $\quad \iint_D \operatorname{curl} \vec{F}\, dA = \int_0^1 \int_0^1 (1-x)\, dy\, dx$

$$= \int_0^1 (1-x)\, dx$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Problem 2. Let $\mathbf{F}(x,y) = \langle x^3, 4x\rangle$, let C be the oriented curve from $A = (-1,0)$ to $B = (-1,-1)$. See the image below. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ using Green's Theorem. *Warning: C is not a closed curve. How can you get around this issue?*



Let D be the oriented line segment from B to A and let R be the region enclosed by C and D.

Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} + \int_D \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA$

Note $\vec{r}(t) = \langle -1, t\rangle$, $-1 \le t \le 0$ parametrizes D and

$$\int_D \vec{F}\cdot d\vec{r} = \int_{-1}^{0} \langle -1,-4\rangle \cdot \langle 0,1\rangle\, dt$$

$$= \int_{-1}^{0} -4\, dt = -4.$$

Further, $\text{curl}\,\vec{F} = 4$, so

$$\iint_R \text{curl}\,\vec{F}\, dA = 4\,\text{area}(R) = 4(4) = 16.$$

Therefore $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA - \int_D \vec{F}\cdot d\vec{r} = 20$

Problem 3. Let $\mathbf{F}(x,y) = \langle 2xe^y, x + x^2e^y \rangle$, $\mathbf{G} = \langle 0, x \rangle$, and let C be the quarter circle oriented from $A = (4,0)$ to $B = (0,4)$. The image below shows $\mathbf{F}$ along with C.

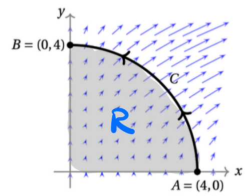
a. Explain why $\mathbf{F}$ does not have a potential function.

b. Find a function f such that $\mathbf{F} = \mathbf{G} + \nabla f$.

c. Let C_1 be the line segment from $(0,0)$ to A and let C_2 be the line segment from $(0,0)$ to B. Find $\int_{C_1} \mathbf{G} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{G} \cdot d\mathbf{r}$. These integrals can be done without computation and instead just thinking about how the vectors of $\mathbf{G}$ are related to the tangent vectors along C_1 and C_2.

d. Use Green's Theorem and part c. to compute $\int_C \mathbf{G} \cdot d\mathbf{r}$.

e. Use parts b. and d. to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.



(a) Note $\text{curl}\,\vec{F} = 1$. Since $\text{curl}\,\vec{F} \neq 0$, $\vec{F}$ is not conservative by the curl test.

(b) $\vec{F} - \vec{G} = \langle 2xe^y, x^2e^y \rangle$. If $\nabla f = \vec{F} - \vec{G}$,

then $f_x = 2xe^y$, which implies $f(x,y) = x^2e^y + C(y)$.

Further, $f_y = x^2e^y$ implies $x^2e^y = \frac{\partial}{\partial y}\left(x^2e^y + C(y)\right)$

which implies $C'(y) = 0$, so $C(y)$ is a constant.

Therefore $f(x,y) = x^2e^y$.

(c) $\int_{C_1} \vec{G} \cdot d\vec{r} = 0$ since $\vec{G}$ is orthogonal to C_1

$\int_{C_2} \vec{G} \cdot d\vec{r} = 0$ since $\vec{G} = \langle 0, 0 \rangle$ along C_2.

(d) Let R be the region enclosed by C, C_1, and C_2.

Note $\operatorname{curl} \vec{G} = 1$. Then

$$\int_{C_1} \vec{G} \cdot d\vec{r} + \int_{C} \vec{G} \cdot d\vec{r} - \int_{C_2} \vec{G} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{G} \, dA$$

$$= \text{area}(R)$$

$$= \frac{1}{4}\pi(4)^2$$

$$= 4\pi$$

Thus $\int_C \vec{G} \cdot d\vec{r} = 4\pi$.

(e) Observe that

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r} + \int_C \nabla f \cdot d\vec{r}$$

$$= 4\pi + f(B) - f(A)$$

$$= 4\pi - 16.$$