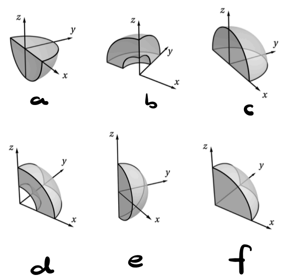
Problem 1. The figures below show different solid regions, either portions of the unit ball or portions of the solid between spheres of radius 1 and 2. Set up triple integrals for the volume of each solid using spherical coordinates.



(a) $$\int_{\theta=0}^{\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(b) $$\int_{\theta=\frac{\pi}{2}}^{\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=1}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(c) $$\int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(d) $$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=1}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(e) $$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(f) $$\int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Problem 2. Convert the following triple integrals from cylindrical to Cartesian coordinates or vice versa.

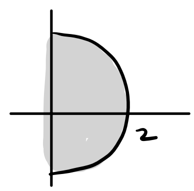
a. $\int_{-\pi/2}^{\pi/2}\int_0^2\int_0^{r^2} r\,dz\,dr\,d\theta$

b. $\int_0^{2\pi}\int_0^2\int_0^r r\,dz\,dr\,d\theta$

c. $\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^2 xz\,dz\,dx\,dy$

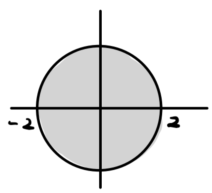
d. $\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

(a)



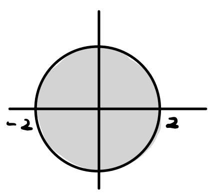
$$\int_{x=0}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=x^2+y^2} dz\,dy\,dx$$

(b)



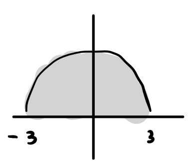
$$\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{\sqrt{x^2+y^2}} dz\,dy\,dx$$

(c)



$$\int_0^{2\pi}\int_0^2\int_r^2 z r^2\cos\theta\,dz\,dr\,d\theta$$

(d)



$$\int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

Problem 3. Convert the following double integrals from Cartesian to polar coordinates.

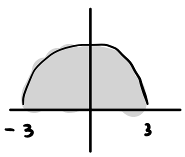
a. $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\, dydx$

b. $\int_{0}^{1} \int_{y}^{\sqrt{2-y^2}} (x+y)\, dxdy$

c. $\int_{0}^{9} \int_{-\sqrt{81-y^2}}^{0} x^2 y\, dxdy$

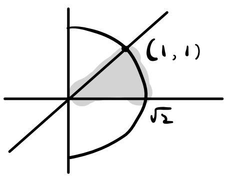
d. $\int_{0}^{2} \int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dydx$

(a)



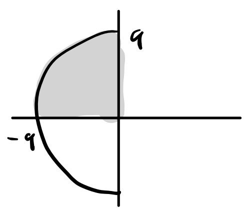
$$\int_0^{\pi} \int_0^{3} r\sin(r^2)\, dr\, d\theta$$

(b)



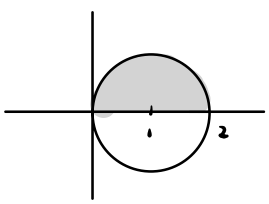
$$\int_0^{\pi/4} \int_0^{\sqrt{2}} r^2(\cos\theta + \sin\theta)\, dr\, d\theta$$

(c)



$$\int_{\pi/2}^{\pi} \int_0^{9} r^4 \cos^2\theta \sin\theta\, dr\, d\theta$$

(d)



$$\int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta$$

$y = \sqrt{2x - x^2}$

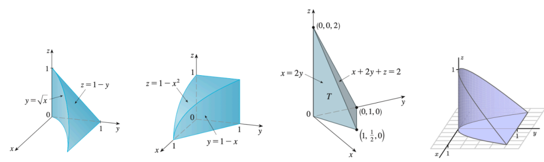
$y^2 = 2x - x^2$

$x^2 - 2x + y^2 = 0 \Rightarrow r^2 = 2r\cos\theta$

$x^2 - 2x + 1 + y^2 = 1 \qquad r = 2\cos\theta$

$(x-1)^2 + y^2 = 1$

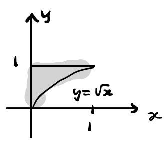
Problem 4. Setup triple integrals for each of the solids below using the following orders of integration: (1) $dV = dz\,dy\,dx$, (2) $dV = dy\,dz\,dx$, (3) $dV = dx\,dz\,dy$. Note the last solid is bounded by $y = x^2$, $z = 1 - y$, and $z = 0$.



a b c d

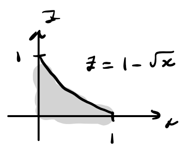
(a)

(1)



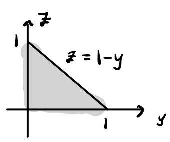
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\,dy\,dx$$

(2)



$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} dy\,dz\,dx$$

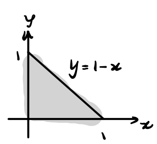
(3)



$$\int_0^1 \int_0^{1-y} \int_0^{y^2} dx\,dz\,dy$$

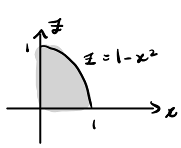
(b)

(1)



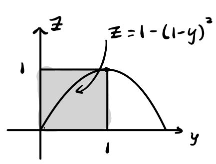
$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} dz\,dy\,dx$$

(2)



$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} dy\,dz\,dx$$

(3)

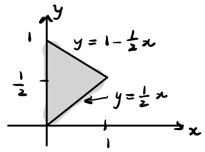


$$\int_0^1 \int_{1-(1-y)^2}^1 \int_0^{\sqrt{1-z}} dx\,dz\,dy + \int_0^1 \int_0^{1-(1-y)^2} \int_0^{1-y} dx\,dz\,dy$$

$z = 1 - x^2$, $x = 1 - y \Rightarrow z = 1 - (1-y)^2$

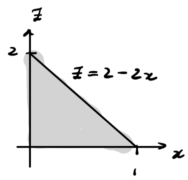
(c)

(1)



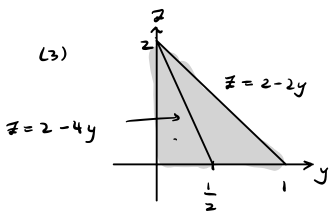
$$\int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \int_0^{2-x-2y} dz\,dy\,dx$$

(2)



$$\int_0^1 \int_0^{2-2x} \int_{x/2}^{\frac{1}{2}(2-x-z)} dy\,dz\,dx$$

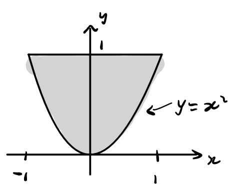
(3)



$$\int_0^{1/2} \int_0^{2-4y} \int_0^{2y} dx\,dz\,dy + \underbrace{\int_{z=0}^{z=2} \int_{y=\frac{1}{4}(2-z)}^{y=\frac{1}{2}(2-z)} \int_0^{2-2y-z} dx\,dy\,dz}_{\text{easier to set up with } dy\,dz \text{ on outside}}$$

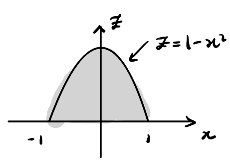
(d)

(1)



$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\,dy\,dx$$

(2)

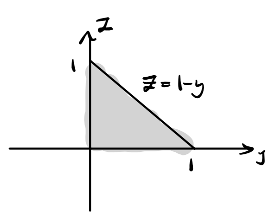


$$\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy\,dz\,dx$$

$y = x^2, \; z = 1 - y$

$\Rightarrow \; 1 - z = x^2 \Rightarrow z = 1 - x^2$

(3)



$$\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$$

Problem 5. For each vector field $\mathbf{F}$ below, determine whether it is conservative. If it is, find a potential function f and use it to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the given curve C. If it is not, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using a parametrization of C.

a. $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$, C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$

b. $\mathbf{F}(x, y) = \langle ye^x + \sin y, e^x + x\cos y \rangle$, C is the quarter of the ellipse $x^2 + 2y^2 = 4$ from $(2, 0)$ to $(0, \sqrt{2})$

c. $\mathbf{F}(x, y) = \langle 2x - 2y, -3x + 4y - 8 \rangle$, C is the quarter of the unit circle from $(0, -1)$ to $(1, 0)$

d. $\mathbf{F}(x, y) = \langle ye^x + \sin y, e^x + x\cos y \rangle$, C is the unit circle oriented counterclockwise

e. $\mathbf{F}(x, y) = \langle xy^2, 2x^2y \rangle$, C is the line segment from $(1, 2)$ to $(3, 4)$

(a) $\text{curl}\,\vec{F} = 0 \Rightarrow$ conservative

$$f_x = x^2 \Rightarrow f(x,y) = \frac{1}{3}x^3 + C(y)$$

$$f_y = y^2 \Rightarrow y^2 = C'(y) \Rightarrow C(y) = \frac{1}{3}y^3 + c$$

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$$

$$\int_C \vec{F} \cdot d\vec{r} = f(2,8) - f(-1,2) = \frac{1}{3}(2)^3 + \frac{1}{3}(8)^3 - \frac{1}{3}(-1)^3 - \frac{1}{3}(2)^3$$

$$= 171$$

(b) $\text{curl}\,\vec{F} = (e^x + \cos y) - (e^x + \cos y) = 0 \Rightarrow$ conservative

$$f_x = ye^x + \sin y \Rightarrow f(x,y) = ye^x + x\sin y + C(y)$$

$$f_y = e^x + x\cos y \Rightarrow e^x + x\cos y = e^x + x\sin y + C'(y)$$

$$\Rightarrow C(y) = c$$

$$\Rightarrow f(x,y) = ye^x + x\sin y$$

$$\int_C \vec{F} \cdot d\vec{r} = f(0,\sqrt{2}) - f(2,0) = \sqrt{2}$$

(c) $\text{curl}\,\vec{F} = -3-(-2) = -1 \neq 0 \Rightarrow$ not conservative

$$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad \frac{3\pi}{2} \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{3\pi/2}^{2\pi} \langle 2\cos t - 2\sin t, -3\cos t + 4\sin t - 8 \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_{3\pi/2}^{2\pi} (-2\sin t\cos t + 2\sin^2 t - 3\cos^2 t + 4\sin t\cos t - 8\cos t)\,dt$$

$$= \int_{3\pi/2}^{2\pi} (2\sin t\cos t + 2 - 5\cos^2 t - 8\cos t)\,dt$$

$$= 2\int_{-1}^{0} u\,du + \pi - 5\int_{3\pi/2}^{2\pi} \frac{1+\cos 2t}{2} - 8\sin t \Big|_{3\pi/2}^{2\pi}$$

$$= -1 + \pi - 5\left(\frac{t}{2} + \frac{1}{4}\sin(2t)\Big|_{3\pi/2}^{2\pi}\right) - 8$$

$$= -1 + \pi - \frac{5\pi}{4} - 8 = -\frac{\pi}{4} - 9$$

(d) (see part b) conservative, $f(x,y) = ye^x + x\sin y$

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

(e) $\text{curl}\,\vec{F} = 4xy - 2xy = 2xy \neq 0 \rightarrow$ not conservative

$\vec{r}(t) = \langle 1,2\rangle + t\langle 2,2\rangle, \quad 0 \le t \le 1$

$= \langle 1+2t, 2+2t\rangle$

$\vec{r}\,'(t) = \langle 2,2\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle (1+2t)(2+2t)^2, 2(1+2t)^2(2+2t)\rangle \cdot \langle 2,2\rangle\, dt$$

$$= \int_0^1 \left(2(1+2t)(2+2t)^2 + 4(1+2t)^2(2+2t)\right) dt$$

$$= \int_0^1 \left(8(1+2t)(1+2t+t^2) + 8(1+4t+4t^2)(1+t)\right) dt$$

$$= \int_0^1 \Big(8(1+2t+t^2+2t+4t^2+2t^3)$$
$$+ 8(1+4t+4t^2+t+4t^2+4t^3)\Big)\, dt$$

$$= 8\int_0^1 (2+9t+13t^2+6t^3)\,dt$$

$$= 8\left(2+\frac{9}{2}+\frac{13}{3}+\frac{3}{2}\right)$$

$$= 8\left(8+\frac{13}{3}\right)$$

$$= 8\left(\frac{37}{3}\right) = \frac{296}{3}$$

Problem 6. For each vector field $\mathbf{F}$ and oriented curve C given below, use Green's Theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Be careful: check the orientation of the given curve when applying the theorem.

a. $\mathbf{F}(x, y) = \langle xy^2, 2x^2y \rangle$, C is the triangle oriented from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$

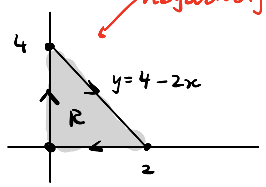
b. $\mathbf{F}(x, y) = \langle \cos y, x^2 \sin y \rangle$, C is the rectangle oriented from $(0,0)$ to $(5,0)$ to $(5,2)$ to $(0,2)$

c. $\mathbf{F}(x, y) = \left\langle y + e^{\sqrt{x}}, 2x + \cos y^2 \right\rangle$, C is the piecewise curve from $(0,0)$ to $(1,1)$ along $x = y^2$ and from $(1,1)$ to $(0,0)$ along $y = x^2$

d. $\mathbf{F}(x, y) = \langle \sin^3 x + 4y, 5x + \cos^2 y \rangle$, C is the circle $(x-3)^2 + (y+4)^2 = 4$ traced clockwise

(a)

negatively oriented



$\text{curl}\,\vec{F} = 2xy$

$$\oint_C \vec{F} \cdot d\vec{r} = -\oint_{-C} \vec{F} \cdot d\vec{r} = -\iint_R \text{curl}\,\vec{F}\, dA$$

$$= -\int_0^2 \int_0^{4-2x} (2xy)\, dy\, dx$$

$$= -\int_0^2 xy^2 \Big|_0^{4-2x} dx$$

$$= -\int_0^2 x(4-2x)^2\, dx$$

$$= -\int_0^2 x(16 - 16x + 4x^2)\, dx$$

$$= -\int_0^2 16x - 16x^2 + 4x^3$$

$$= -\left(8x^2 - \frac{16}{3}x^3 + x^4 \Big|_0^2 \right)$$

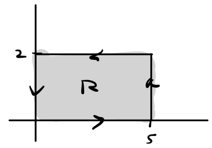
$$= -\left(32 - \frac{128}{3} + 16 \right)$$

$$= -\left(\frac{96 - 128 + 48}{3} \right)$$

$$= -\frac{16}{3}$$

(b) $\text{curl}\,\vec{F} = 2x\sin y + \sin y$

$= \sin y\,(2x+1)$



$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$$

$$= \int_0^5 \int_0^2 \sin y\,(2x+1)\,dy\,dx$$

$$= \int_0^5 (2x+1)\cos y\Big|_2^0\,dx$$

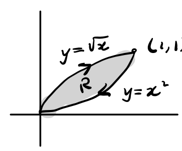
$$= (1-\cos 2)\int_0^5 (2x+1)\,dx$$

$$= (1-\cos 2)\left(x^2+x\Big|_0^5\right)$$

$$= 30(1-\cos 2)$$

(c) $\text{curl}\,\vec{F} = 2-1 = 1$

negatively oriented



$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

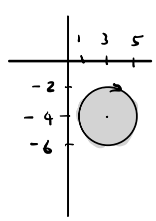
$$= -\iint_R \text{curl}\,\vec{F}\,dA$$

$$= -\int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx$$

$$= -\int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{-1}{3}$$

(d) $\text{curl}\,\vec{F} = 5-4 = 1$



$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

$$= -\iint_R \text{curl}\,\vec{F}\,dA$$

$$= -\text{area}(R)$$

$$= -\pi(2)^2$$

$$= -4\pi$$

Problem 7. For each vector field $\mathbf{F}$ and oriented curve C given below, set up $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ using a parametrization of C. If possible, use the Divergence Theorem to compute $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$. Otherwise compute it using your parametrization

a. $\mathbf{F}(x,y) = \langle y^3, x^2\rangle$, C is the circle $x^2+y^2=9$ traced counterclockwise

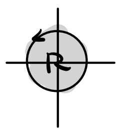
b. $\mathbf{F}(x,y) = \langle \cos x, \sin y\rangle$, C is line segment from $(2,0)$ to $(0,2)$

c. $\mathbf{F}(x,y) = \langle x^2, y\rangle$, C is the piecewise closed curve from $(0,1)$ to $(1,0)$ along $y=1-x^2$, to $(0,0)$ along $y=0$, to $(0,1)$ along $x=0$

d. $\mathbf{F}(x,y) = \langle 0, x\rangle$, C is the line segment from $(1,1)$ to $(5,1)$

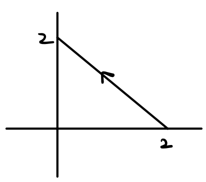
(a) $\operatorname{div}\vec{F} = 0$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$$



$$= 0$$

(b) C not closed $\Rightarrow$ Divergence Theorem not applicable



$$\vec{r}(t) = \langle 2,0\rangle + t\langle -2,2\rangle,\quad 0\le t\le 1$$

$$= \langle 2-2t, 2t\rangle$$

$$\vec{r}\,'(t) = \langle -2, 2\rangle$$

$$\int_C \vec{F}\cdot\vec{n}\,ds$$

$$= \int_0^1 \langle \cos(2-2t), \sin(2t)\rangle \cdot \langle 2,2\rangle\,dt$$

$$= 2\int_0^1 (\cos(2-2t) + \sin 2t)\,dt$$

$$= 2\left(-\frac{1}{2}\int_2^0 \cos u\,du + \frac{1}{2}\int_0^2 \sin u\,du\right)$$

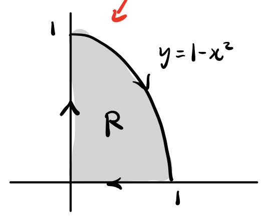
$$= \int_0^2 (\cos u + \sin u)\,du$$

$$= \sin u - \cos u \Big|_0^2$$

$$= \sin 2 - \cos 2 - (\sin 0 - \cos 0) = \sin 2 - \cos 2 + 1$$

negatively oriented

(c)



$$\operatorname{div}\vec{F} = 2x+1$$

$$-\iint_R \operatorname{div}\vec{F}\, dA$$

$$= -\int_0^1\int_0^{1-x^2} (2x+1)\, dy\, dx$$

$$= -\int_0^1 (2x+1)(1-x^2)\, dx$$

$$= -\int_0^1 (1+2x-x^2-2x^3)\, dx$$

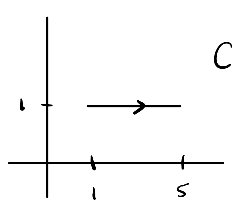
$$= -\left(x+x^2-\tfrac{1}{3}x^3-\tfrac{1}{2}x^4\Big|_0^1\right)$$

$$= -\left(1+1-\tfrac{1}{3}-\tfrac{1}{2}\right)$$

$$= -\left(\tfrac{12}{6}-\tfrac{2}{6}-\tfrac{3}{6}\right)$$

$$= -\frac{7}{6}$$

(d)



C not closed $\Rightarrow$ Divergence Theorem not applicable

$$\vec{F}(t) = \langle t, 1\rangle, \quad 1 \le t \le 5$$

$$\vec{F}'(t) = \langle 1, 0\rangle$$

$$\int_1^5 \langle 0, t\rangle \cdot \langle 0, -1\rangle\, dt$$

$$= \int_1^5 -t\, dt = -\tfrac{1}{2}t^2\Big|_1^5 = \tfrac{1}{2}(1-25) = -12$$