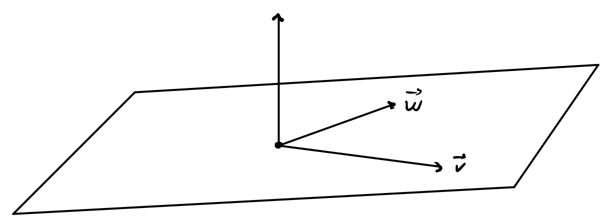
§10.4 Cross Product

Goal Given $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, form a new vector that is orthogonal to both.



plane containing $\vec{v}$ and $\vec{w}$ and a vector orthogonal to both

Preliminary notation $\vec{i}, \vec{j}, \vec{k} \in \mathbb{R}^3$ are the vectors

$$\vec{i} = \langle 1, 0, 0 \rangle, \quad \vec{j} = \langle 0, 1, 0 \rangle, \quad \vec{k} = \langle 0, 0, 1 \rangle$$

Definition Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Their cross product is the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_1 \rangle$$

Computational method

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

bars mean take determinant of the matrix

alternate signs

$$= \vec{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \vec{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \vec{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \vec{i}\left(v_2 w_3 - v_3 w_2\right) - \vec{j}\left(v_1 w_3 - v_3 w_1\right) + \vec{k}\left(v_1 w_2 - v_2 w_1\right)$$

$$= \langle v_2 w_3 - v_3 w_2, -\left(v_1 w_3 - v_3 w_1\right), v_1 w_2 - v_2 w_1 \rangle$$

Example Let $\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{w} = \langle 4, 5, 6 \rangle$

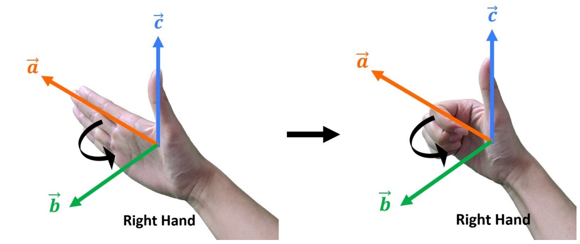
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \vec{i}\,(12 - 15) - \vec{j}\,(6 - 12) + \vec{k}\,(5 - 8)$$

$$= \langle -3, 6, -3 \rangle$$

Properties

① <u>Right hand rule</u> the vector $\vec{v} \times \vec{w}$ points in the direction your thumb points if you curl your fingers from $\vec{v}$ to $\vec{w}$ with your right hand.



② $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

③ $\vec{v} \times \vec{v} = \vec{0}$

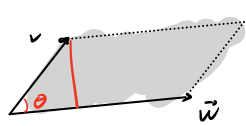
④ $(\vec{v} \times \vec{w}) \cdot \vec{v} = (\vec{v} \times \vec{w}) \cdot \vec{w} = 0$

⑤ $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$.

this can be derived from the identity
$\|\vec{v} \times \vec{w}\|^2$
$= \|\vec{v}\|^2 \|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2$

⑥ area of parallelogram formed by $\vec{v}$ and $\vec{w}$ is $\|\vec{v} \times \vec{w}\|$

area of parallelogram
$= \text{base} \times \text{height}$
$= (\|\vec{w}\|)(\|\vec{v}\| \sin\theta)$

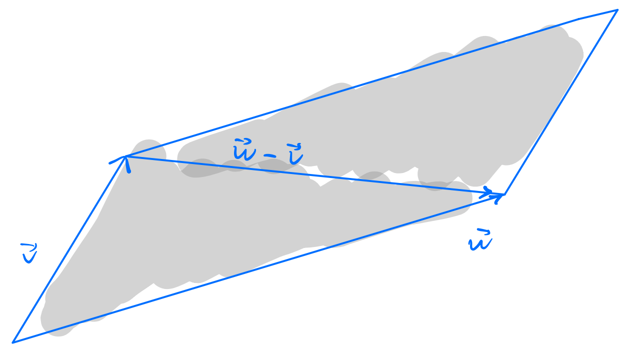


← area is $\|\vec{v} \times \vec{w}\|$

Example Let $\vec{v} = \langle 1, 1, 2 \rangle$, $\vec{w} = \langle -1, -3, -1 \rangle$

Find ① area of parallelogram formed by $\vec{v}, \vec{w}$

② area of triangle formed by $\vec{v}, \vec{w}, \vec{w}-\vec{v}$



①
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -1 & -3 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-1+6) - \vec{j}\,(-1+2) + \vec{k}\,(-3+1)$$

$$= \langle 5, -1, -2 \rangle$$

$$\| \vec{v} \times \vec{w} \| = \sqrt{25+1+4} = \sqrt{30}$$

② $\frac{1}{2} \| \vec{v} \times \vec{w} \| = \frac{1}{2}\sqrt{30}$

Problem 1. Compute $\mathbf{v} \times \mathbf{w}$ for each example below.

a. $\mathbf{v} = \langle 3, 2, -2 \rangle$, $\mathbf{w} = \langle 0, 1, 5 \rangle$

b. $\mathbf{v} = \langle 4, -5, -5 \rangle$, $\mathbf{w} = \langle 3, 3, 4 \rangle$

c. $\mathbf{v} = \mathbf{i}, \mathbf{w} = \mathbf{j}$

d. $\mathbf{v} = \mathbf{i}, \mathbf{w} = \mathbf{k}$

Problem 2. Without doing additional computation, find $\mathbf{w} \times \mathbf{v}$ in each example of Problem 1.

(a)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -2 \\ 0 & 1 & 5 \end{vmatrix} = \langle 12, -15, 3 \rangle$$

$$\vec{w} \times \vec{v} = \langle -12, 15, -3 \rangle$$

(b)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -5 & -5 \\ 3 & 3 & 4 \end{vmatrix} = \langle -5, -31, 27 \rangle$$

$$\vec{w} \times \vec{v} = \langle 5, 35, -27 \rangle$$

(c)

$$\vec{i} \times \vec{j} = \vec{k} = \langle 0, 0, 1 \rangle$$

$$\vec{j} \times \vec{i} = -\vec{k} = \langle 0, 0, -1 \rangle$$

(d)

$$\vec{i} \times \vec{k} = -\vec{j} = \langle 0, -1, 0 \rangle$$

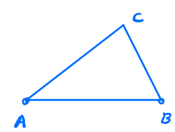
$$\vec{k} \times \vec{i} = \vec{j} = \langle 0, 1, 0 \rangle$$

Problem 3. Find the area of the parallelogram formed by $\mathbf{v} = \langle 1,1,1 \rangle$, $\mathbf{w} = \langle 5,0,7 \rangle$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 5 & 0 & 7 \end{vmatrix} = \langle 7, -2, -5 \rangle$$

$$\| \vec{v} \times \vec{w} \| = \sqrt{49 + 4 + 25} = \sqrt{78}$$

Problem 4. Find the area of the triangle with vertices $A = (0,1,0)$, $B = (1,3,-1)$, and $C = (2,1,1)$.



$$\vec{v} = \vec{AB}$$
$$= \langle 1, 2, -1 \rangle$$

$$\vec{w} = \vec{AC}$$
$$= \langle 2, 0, 1 \rangle$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$$
$$= \langle 2, -3, 4 \rangle$$

$$\frac{1}{2} \| \vec{v} \times \vec{w} \|$$
$$= \frac{1}{2} \sqrt{4 + 9 + 16}$$
$$= \frac{1}{2} \sqrt{29}$$