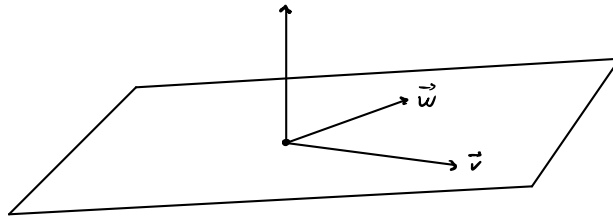## § 10.4 Cross Product

Goal   Given $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, form

a new vector that is orthogonal to both.

plane containing $\vec{v}$ and $\vec{w}$
and a vector orthogonal to both

Preliminary notation $\vec{i}, \vec{j}, \vec{k} \in \mathbb{R}^3$ are the vectors

$$\vec{i} = \langle 1, 0, 0 \rangle, \quad \vec{j} = \langle 0, 1, 0 \rangle, \quad \vec{k} = \langle 0, 0, 1 \rangle$$

Definition   Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Their cross product is the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, \; -(v_1 w_3 - v_3 w_1), \; v_1 w_2 - v_2 w_1 \rangle$$
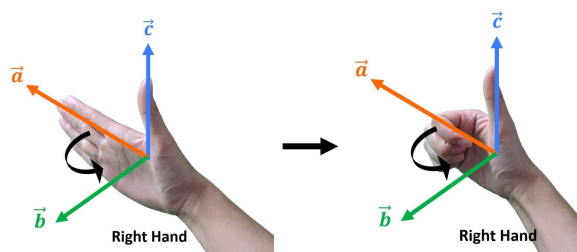
Computational method

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

bars mean take determinant of the matrix

$$= \vec{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \vec{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \vec{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

alternate signs

$$= \vec{i} \left(v_2 w_3 - v_3 w_2 \right) - \vec{j} \left(v_1 w_3 - v_3 w_1 \right) + \vec{k} \left(v_1 w_2 - v_2 w_1 \right)$$

$$= \left\langle v_2 w_3 - v_3 w_2 , -\left(v_1 w_3 - v_3 w_1 \right), v_1 w_2 - v_2 w_1 \right\rangle$$

Example   Let $\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{w} = \langle 4, 5, 6 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \vec{i} \left(12 - 15 \right) - \vec{j} \left(6 - 12 \right) + \vec{k} \left(5 - 8 \right)$$

$$= \langle -3, 6, -3 \rangle$$

Properties

① Right hand rule   the vector $\vec{V} \times \vec{W}$ points

in the direction your thumb points if you

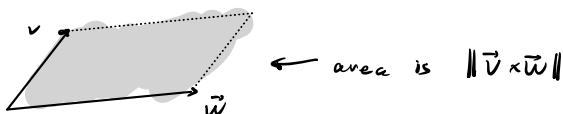curl your fingers from $\vec{V}$ to $\vec{W}$ with your

right hand.

② $\vec{V} \times \vec{W} = -\vec{W} \times \vec{V}$

③ $\vec{V} \times \vec{V} = \vec{0}$

④ $(\vec{V} \times \vec{W}) \cdot \vec{V} = (\vec{V} \times \vec{W}) \cdot \vec{W} = 0$

⑤ $\| \vec{V} \times \vec{W} \| = \| \vec{V} \| \| \vec{W} \| \sin\Theta$ where Θ is
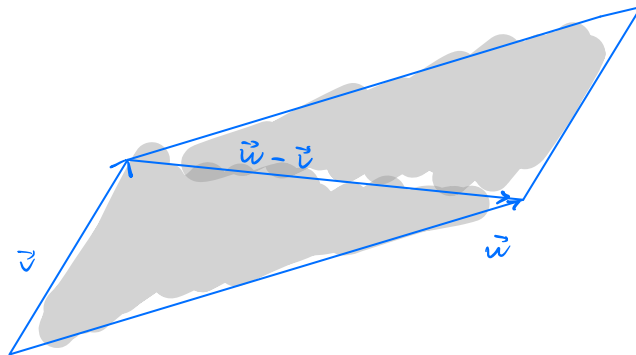
the angle between $\vec{V}$ and $\vec{W}$.

⑥ area of parallelogram formed by $\vec{V}$ and $\vec{W}$

is $\| \vec{V} \times \vec{W} \|$

← area is $\| \vec{V} \times \vec{W} \|$

Example   Let $\vec{v} = \langle 1, 1, 2 \rangle$, $\vec{w} = \langle -1, -3, -1 \rangle$

Find ① area of parallelogram formed by $\vec{v}, \vec{w}$

② area of triangle formed by $\vec{v}, \vec{w}, \vec{w} - \vec{v}$

① $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -1 & -3 & -1 \end{vmatrix}$

$= \vec{i} (-1 + 6) - \vec{j} (-1 + 2) + \vec{k} (-3 + 1)$

$= \langle 5, -1, -2 \rangle$

$\| \vec{v} \times \vec{w} \| = \sqrt{25 + 1 + 4} = \sqrt{30}$

② $\frac{1}{2} \| \vec{v} \times \vec{w} \| = \frac{1}{2} \sqrt{30}$

Problem 1. Compute $\mathbf{v} \times \mathbf{w}$ for each example below.

a. $\mathbf{v} = \langle 3, 2, -2 \rangle$, $\mathbf{w} = \langle 0, 1, 5 \rangle$

b. $\mathbf{v} = \langle 4, -5, -5 \rangle$, $\mathbf{w} = \langle 3, 3, 4 \rangle$

c. $\mathbf{v} = \mathbf{i}, \mathbf{w} = \mathbf{j}$

d. $\mathbf{v} = \mathbf{i}, \mathbf{w} = \mathbf{k}$

Problem 2. Without doing additional computation, find $\mathbf{w} \times \mathbf{v}$ in each example of Problem 1.

(a)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 2 & -2 \\ 0 & 1 & 5 \end{vmatrix} = \langle 12, -15, 3 \rangle$$

$$\vec{w} \times \vec{v} = \langle -12, 15, -3 \rangle$$

(b)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4 & -5 & -5 \\ 3 & 3 & 4 \end{vmatrix} = \langle -5, -31, 27 \rangle$$

$$\vec{w} \times \vec{v} = \langle 5, 35, -27 \rangle$$

(c) $\quad \vec{\imath} \times \vec{\jmath} = \vec{k} = \langle 0, 0, 1 \rangle$

$\vec{\jmath} \times \vec{\imath} = -\vec{k} = \langle 0, 0, -1 \rangle$

(d) $\quad \vec{\imath} \times \vec{k} = -\vec{\jmath} = \langle 0, -1, 0 \rangle$

$\vec{k} \times \vec{\imath} = \vec{\jmath} = \langle 0, 1, 0 \rangle$

Problem 3. Find the area of the parallelogram formed by $\mathbf{v} = \langle 1,1,1 \rangle$, $\mathbf{w} = \langle 5,0,7 \rangle$.
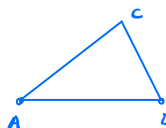
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 5 & 0 & 7 \end{vmatrix} = \langle 7, -2, -5 \rangle$$

$$\| \vec{v} \times \vec{w} \| = \sqrt{49 + 4 + 25} = \sqrt{78}$$

Problem 4. Find the area of the triangle with vertices $A = (0,1,0)$, $B = (1,3,-1)$, and $C = (2,1,1)$.

$$\vec{v} = \vec{AB}$$
$$= \langle 1, 2, -1 \rangle$$

$$\vec{w} = \vec{AC}$$
$$= \langle 2, 0, 1 \rangle$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \langle 2, -3, 4 \rangle$$

$$\frac{1}{2} \| \vec{v} \times \vec{w} \|$$

$$= \frac{1}{2} \sqrt{4 + 9 + 16}$$

$$= \frac{1}{2} \sqrt{29}$$