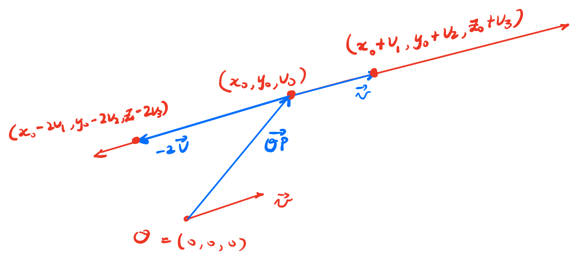
10.5 Lines

<u>Main</u> <u>idea</u> every Line in 3d space is determined by a point that it passes through and a direction vector it's parallel to.

<u>Goal</u> Given a point $P = (x_0, y_0, z_0)$ and a direction $\vec{v} = \langle v_1, v_2, v_3 \rangle$ write an equation for the line passing through P in direction of $\vec{v}$.



<u>Vector equation of the line</u>

for every $t \in \mathbb{R}$, let

$$\ell(t) = \vec{OP} + t\vec{v}$$

$$= \langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle$$

$$= \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$$

the components of this vector are the components of a point on the line.

<u>Parametric equations</u> Equivalently, we can express the line as all points (x, y, z) where

$$x = x_0 + tv_1$$
$$y = y_0 + tv_2$$
$$z = z_0 + tv_3$$

where t ranges over all real numbers

Example Write vector equation of the line passing through $P = (1,2,3)$ and $Q = (1,4,-1)$.

Does the line pass through $(1,8,-5)$? $(1,12,4)$? $(0,0,0)$?

$$\vec{v} = \overrightarrow{PQ} = \langle 0, 2, -4 \rangle$$

$$\ell(t) = \langle 1,2,3 \rangle + t\langle 0,2,-4 \rangle$$

$$= \langle 1, 2+2t, 3-4t \rangle$$

Are there values of t so that

$\ell(t) = \langle 1,8,-5 \rangle$ or $\langle 1,12,4 \rangle$ or $\langle 0,0,0 \rangle$?

$$\begin{cases} 1=1 \\ 2+2t=8 \Rightarrow t=3 \\ 3-4t=-5 \Rightarrow 3-4(3) \neq -5 \end{cases}$$

$\times$

no

$$\begin{cases} 1=1 \\ 2+2t=12 \Rightarrow t=5 \\ 3-4t=4 \Rightarrow 3-4(5) \neq 4 \end{cases}$$

$\times$

no

$$\begin{cases} 1=0 \\ 2+2t=0 \\ 3-4t=0 \end{cases}$$

$\times$

no

In 3d, there are 4 possibilities for the spatial relationship between two lines:

1. parallel (and share 0 points)
2. are the same line (share all points)
3. intersect (share 1 point)
4. skew (share 0 points, not parallel)

Example Consider the lines

$\ell_1(t) = \langle 1,1,1\rangle + t\langle 2,3,4\rangle$

$\ell_2(t) = \langle 11,16,21\rangle + t\langle -4,-6,-8\rangle$

Which option is true about ℓ_1 and ℓ_2?

Direction vectors:

$\vec{v}_1 = \langle 2,3,4\rangle$

$\vec{v}_2 = \langle -4,-6,-8\rangle$

Since $\vec{v}_1 = -\frac{1}{2}\vec{v}_2$, the lines are parallel.

Same line? Notice $(1,1,1)$ lies on ℓ_1. Does it lie on ℓ_2?

Can we find t so that $\langle 11,16,21\rangle + t\langle -4,-6,-8\rangle = \langle 1,1,1\rangle$?

Need single value of t that satisfies all 3 eq's

$$\begin{cases} 11-4t = 1 \\ 16-6t = 1 \\ 21-8t = 1 \end{cases} \Rightarrow t = \frac{5}{2} \quad \text{yes!}$$

Since they're parallel and share 1 point, they must share all points and be the same line.

Example $\ell_1(t) = \langle 1+t, -2+3t, 4-t \rangle$

$\ell_2(t) = \langle 2t, 3+t, -3+4t \rangle$

Same question as previously

Direction vectors:

$\vec{v}_1 = \langle 1, 3, -1 \rangle$

$\vec{v}_2 = \langle 2, 1, 4 \rangle$

Are they parallel? Can we find single c so that $\langle 1,3,-1 \rangle = c \langle 2,1,4 \rangle$?

$$\begin{cases} 1 = 2c \Rightarrow c = \frac{1}{2} \\ 3 = c \Rightarrow c = 3 \\ -1 = 4c \Rightarrow c = -\frac{1}{4} \end{cases}$$

× Not parallel.

Do they intersect? Can we find t, s so that

$$\begin{cases} 1+t = 2s & (1) \\ -2+3t = 3+s & (2) \\ 4-t = -3+4s & (3) \end{cases}$$

Use (1) and (2) to solve for s and t. Then plug into (3)

(1) $\Rightarrow$ $t = 2s - 1$

(2) $\Rightarrow$ $-2 + 3(2s-1) = 3+s$

$\Rightarrow$ $5s = 8$, $s = \frac{8}{5}$ $\Rightarrow$ $t = \frac{16}{5} - \frac{5}{5} = \frac{11}{5}$

$s = 8/5$, $t = 11/5$ Do these satisfy (3) too?

(3) $\Rightarrow$ $4 - \frac{11}{5} = -3 + 4\left(\frac{8}{5}\right)$

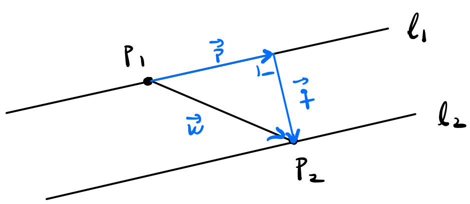
$\frac{20-11}{5} = \frac{-15+40}{5}$ $\frac{9}{5} \neq \frac{25}{5}$ No

So there are no s and t possible. They are skew lines

Example Find the distance between

$$\ell_1(t) = \langle 1,1,1 \rangle + t\langle 2,3,4 \rangle$$

$$\ell_2(t) = \langle 26,2,3 \rangle + t\langle -4,-6,-8 \rangle$$



Let $P_1 = (1,1,1)$ and $P_2 = (26,2,3)$

be points on ℓ_1 and ℓ_2 respectively.

Let $\vec{w} = \overrightarrow{P_1P_2} = \langle 25, 0, 2 \rangle$.

Let $\vec{v} = \langle 2,3,4 \rangle$ be a direction vector for these lines.

Let $\vec{p} = \text{proj}_{\vec{v}}\, \vec{w} = \left(\dfrac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} \right)\vec{v} = \left(\dfrac{50+0+8}{4+9+16} \right)\vec{v}$

$= 2\vec{v} = \langle 4,6,8 \rangle$

and $\vec{q} = \vec{w} - \vec{p} = \langle 25,0,2 \rangle - \langle 4,6,8 \rangle$

$= \langle 21,-6,-6 \rangle$.

The distance is $\|\vec{q}\| = \sqrt{513}$

Problem 1. Write vector equations of the following lines.

a. Line ℓ_1 passing through the points $P = (2, 1, 5), Q = (7. - 2, 4)$

b. Line ℓ_2 passing through the points $P = (1, 2, 1), Q = (11, -4, -1)$

c. Line ℓ_3 passing through $P = (0, 1, 2)$ and orthogonal to both vectors $\mathbf{v} = \langle 1, 2, -1 \rangle, \mathbf{w} = \langle 2, 0, 1 \rangle$.

(a) $\vec{v} = \vec{PQ} = \langle 5, -3, -1 \rangle$

$\ell_1(t) = \langle 2, 1, 5 \rangle + t \langle 5, -3, -1 \rangle$

(b) $\vec{v} = \vec{PQ} = \langle 10, -6, -2 \rangle$

$\ell_2(t) = \langle 1, 2, 1 \rangle + t \langle 10, -6, -2 \rangle$

(c) $\vec{d} = \vec{v} \times \vec{w}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$$

$= \langle 2, -3, 4 \rangle$

$\ell_3(t) = \langle 0, 1, 2 \rangle + t \langle 2, -3, 4 \rangle$

Problem 2. Are any of the lines in Problem 1 parallel? For each pair of lines, explain why or why not.

(a) and (b) are parallel since

$\langle 5, -3, -1 \rangle = \frac{1}{2} \langle 10, -6, -2 \rangle$

(c) is not parallel to either of (a) or (b)

since there doesn't exist a scalar c

such that $\langle 2, -3, 4 \rangle = c \langle 10, -6, -2 \rangle$.

Problem 3. Determine whether the point $(2, 1, 5)$ lies on both ℓ_1 and ℓ_2 and make a conclusion about whether ℓ_1 and ℓ_2 are the same line.

Note $(2,1,5)$ lies on ℓ_1. Does it lie on ℓ_2?

Does there exist t so that

$$\langle 1,2,1\rangle + t\langle 10,-6,-2\rangle = \langle 2,1,5\rangle \ ?$$

$$\begin{cases} 1+10t=2 \Rightarrow t = 1/10 \\ 2-6t=1 \Rightarrow t=1/6 \quad \times \quad \text{No.} \\ 1-2t=5 \Rightarrow t=-2 \end{cases}$$

So ℓ_1 and ℓ_2 are parallel but not the same line.

Problem 4. The following two lines intersect:

$$\ell_1(t) = \langle 5, 0, 3\rangle + t\langle -1, 1, 1\rangle$$
$$\ell_2(t) = \langle 1, 4, 7\rangle + t\langle 3, 0, -3\rangle .$$

Find their point of intersection.

Find s,t so that $\ell_1(t) = \ell_2(s)$:

$$\begin{cases} 5-t = 1+3s & (1) \\ t = 4 & (2) \\ 3+t = 7-3s & (3) \end{cases}$$

plugging (2) into (1) gives

$$1 = 1+3s \Rightarrow s=0$$

So $(1,4,7)$ is the point of intersection.