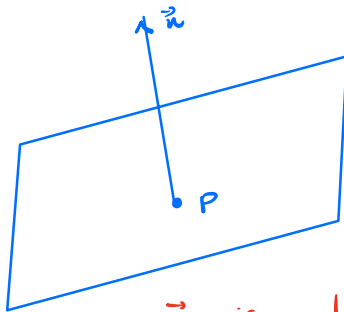


10.6 Planes

Goal Given a point $P = (x_0, y_0, z_0)$
and a vector $\vec{n} = \langle a, b, c \rangle$ write an
equation for the plane passing through P
and orthogonal to \vec{n}



\vec{n} is called a normal vector
for the plane.

Standard form equation The plane consists

of all points $Q = (x, y, z)$ so that

\vec{PQ} and \vec{n} are orthogonal:

$$0 = \vec{PQ} \cdot \vec{n}$$

$$= \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

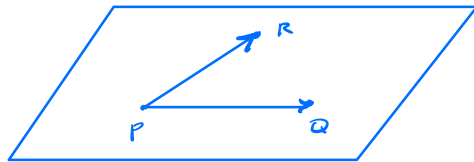
General form equation of plane

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example Let $P = (3, -1, 2)$, $Q = (8, 2, 4)$, $R = (-1, -2, -3)$.

Find standard form and general form equations of the plane containing P, Q, R .



$$\vec{v} = \vec{PQ} = \langle 5, 3, 2 \rangle$$

$$\vec{w} = \vec{PR} = \langle -4, -1, -5 \rangle$$

$$\vec{n} = \vec{v} \times \vec{w}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix}$$

$$= \langle -13, 17, 7 \rangle$$

$$-13(x-3) + 17(y+1) + 7(z-2) = 0 \quad \text{std form}$$

$$-13x + 17y + 7z = -42 \quad \text{general form}$$

Example Consider the plane $2x + y + 3z = 1$.

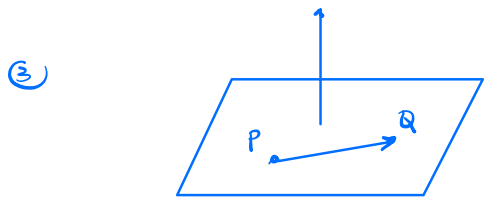
- ① Find a normal vector \vec{n} .
- ② Find a point P on plane
- ③ Find a vector orthogonal to \vec{n} .

① $\vec{n} = \langle 2, 1, 3 \rangle$

② If $x=1, y=2,$

$$2(1) + (2) + 3z = 1 \rightarrow z = -1$$

$$\vec{P} = (1, 2, -1)$$



any vector \vec{PQ} on plane is orthogonal to \vec{n} , so let's find another point Q on the plane: $x=0, y=-2 \rightarrow 2(0) + (-2) + 3z = 1$

$$Q = (0, -2, 1)$$

$$\vec{PQ} = \langle -1, -4, 2 \rangle \text{ works (check}$$

$$\text{that } \vec{PQ} \cdot \vec{n} = 0)$$

Problem 1. Give a standard form equation of the plane containing the points $(1, 2, 3)$, $(3, -1, 4)$ and $(1, 0, 1)$.

$$P = (1, 2, 3), \quad Q = (3, -1, 4), \quad R = (1, 0, 1)$$

$$\vec{PQ} = \langle 2, -3, 1 \rangle, \quad \vec{PR} = \langle 0, -2, -2 \rangle$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 0 & -2 & -2 \end{vmatrix} = \langle 8, 4, -4 \rangle$$

$$8(x-1) + 4(y-2) - 4(z-3) = 0$$

Problem 2. Give a standard form equation of the plane that contains the intersecting lines $\ell_1(t) = \langle 2, 1, 2 \rangle + t \langle 1, 2, 3 \rangle$ and $\ell_2(t) = \langle 2, 1, 2 \rangle + t \langle 2, 5, 4 \rangle$.

$$\vec{v} = \langle 1, 2, 3 \rangle, \quad \vec{w} = \langle 2, 5, 4 \rangle$$

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{vmatrix} = \langle -7, 2, 1 \rangle$$

$$-7(x-2) + 2(y-1) + (z-2) = 0$$

Problem 3. Give a standard form equation of the plane containing the point $(1, 2, 3)$ that is parallel to the plane $2x + 4y + 6z = 8$.

$$\vec{n} = \langle 2, 4, 6 \rangle$$

$$2(x-1) + 4(y-2) + 6(z-3) = 0$$

Problem 4. Consider the line $\ell(t) = \langle 5, 1, -1 \rangle + t \langle 2, 2, 1 \rangle$ and the plane $5x - y - z = -3$.

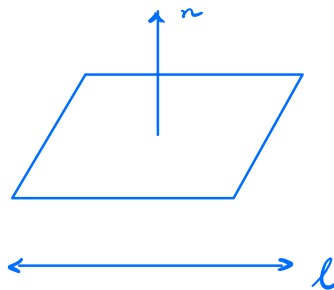
- Compute the dot product between $\mathbf{v} = \langle 2, 2, 1 \rangle$ and a normal vector \mathbf{n} for the plane.
- Use your previous answer to explain why the line and plane are not parallel.
- Since the line and plane are not parallel, they must intersect. Find the point of intersection by finding a value t so that $\ell(t)$ is a point on the plane.

(a) $\vec{n} = \langle 5, -1, -1 \rangle$

$$\vec{n} \cdot \vec{v} = 10 - 2 - 1 = 7 \neq 0$$

(b) If line and plane parallel, \vec{v} and \vec{n} are orthogonal. But

$\vec{v} \cdot \vec{n} \neq 0$ means they're not orthogonal.



(c) Find t so that

$(5 + 2t, 1 + 2t, -1 + t)$ satisfies

$$5x - y - z = -3 :$$

$$5(5 + 2t) - (1 + 2t) - (-1 + t) = -3$$

$$\Rightarrow 25 + 10t - 1 - 2t + 1 - t = -3$$

$$\Rightarrow 7t = -28$$

$$t = -4$$

So $(-3, -7, -5)$ is point of intersection.