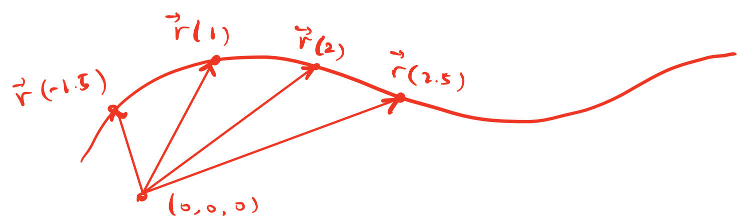
11.1 Vector-valued functions

Def A vector-valued function is a function whose domain is $\mathbb{R}$ and whose output is vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$:

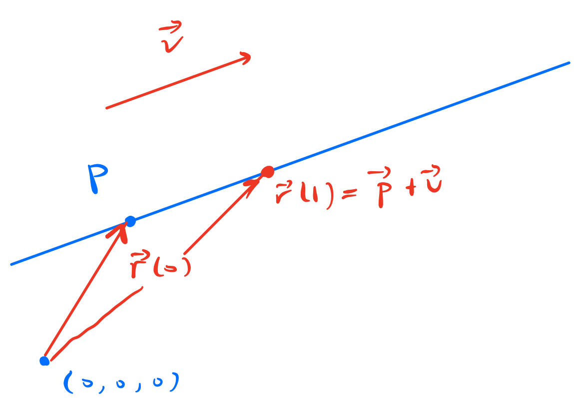
$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle.$$



$\vec{r}(t)$ describes position of particle moving in space

Example A line passing through $P = (x_0, y_0, z_0)$ in direction of $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\vec{r}(t) = \langle x_0 + tv_1,\ y_0 + tv_2,\ z_0 + tv_3 \rangle.$$



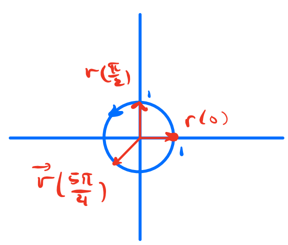
<u>Example</u> Plot the curves described by

$$\vec{r}_1(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle \quad \leftarrow \text{Notice } x^2 + y^2 = 1$$

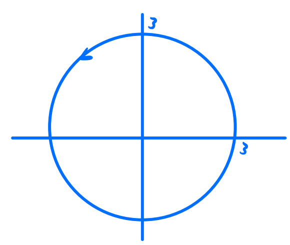
$$\vec{r}_2(t) = \langle 3\cos t, 3\sin t \rangle$$

$$\vec{r}_3(t) = \langle 2 + \cos t, 3 + \sin t \rangle$$

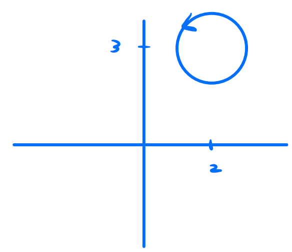
for $0 \le t \le 2\pi$.



unit circle



circle with radius 3, center at (0,0)



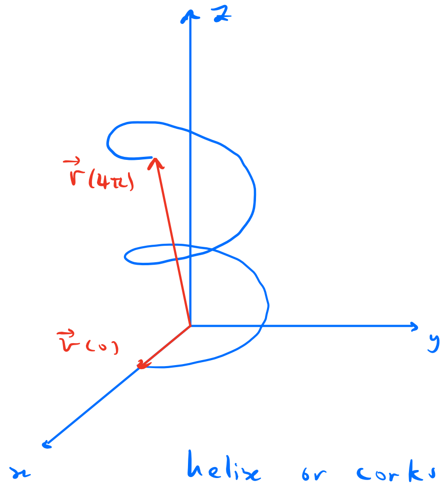
circle with radius 1, center at (2,3)

t	$\vec{r}(t)$
0	$\langle 1, 0 \rangle$
$\pi/4$	$\langle \sqrt{2}/2, \sqrt{2}/2 \rangle$
$\pi/2$	$\langle 0, 1 \rangle$
π	$\langle -1, 0 \rangle$

<u>Circle equation</u> $\vec{r}(t) = \langle a + r\cos t, b + r\sin t \rangle$

describes circle with center (a,b) and radius r traced counter-clockwise.

Example Plot $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 4\pi$.

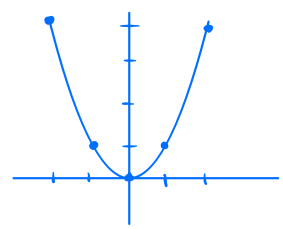


helix or corkscrew

Example Plot $\vec{r}(t) = \langle t, t^2 \rangle$, $-2 \le t \le 2$

We can make a table with example values:

t	$\vec{r}(t)$
-2	$\langle -2, 4 \rangle$
-1	$\langle -1, 1 \rangle$
0	$\langle 0, 0 \rangle$
1	$\langle 1, 1 \rangle$
2	$\langle 2, 4 \rangle$



But notice $x = t$ and $y = t^2 \Rightarrow y = x^2$

General Rule To make a vector-valued function whose plot is the graph of $y = f(x)$ for $a \le x \le b$ let $\vec{r}(t) = \langle t, f(t) \rangle$, $a \le t \le b$.

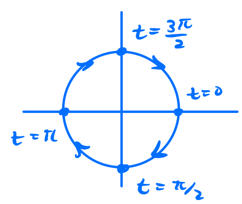
Problem 1. Make a vector equation $\mathbf{r}(t)$ for a circle of radius 2 with center $(-1, 3)$ that is traced counter-clockwise for $0 \le t \le 2\pi$. What point on the curve is given by time $t = 0$? $t = \pi/2$? $t = 2\pi$?

$$\vec{r}(t) = \langle -1, 3 \rangle + 2\langle \cos t, \sin t \rangle$$

$$= \langle -1 + 2\cos t, 3 + 2\sin t \rangle$$

$\vec{r}(0) = \langle 1, 3 \rangle$, $\vec{r}(\pi/2) = \langle -1, 5 \rangle$, $\vec{r}(2\pi) = \langle 1, 3 \rangle$

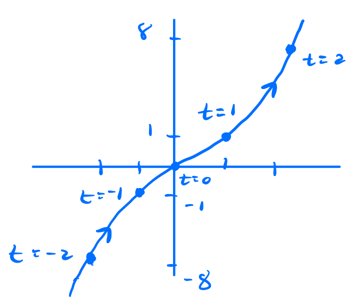
Problem 2. Plot the curve given by $\mathbf{r}(t) = \langle \cos(-t), \sin(-t) \rangle$ for $0 \le t \le 2\pi$ by first plotting some example points using $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$. What curve is it? What effect does the negative sign have?



unit circle, but it's traced clockwise when using $-t$ instead of t.

Problem 3. Plot the curve given by $\mathbf{r}(t) = \langle t, t^3 \rangle$ for $-2 \le t \le 2$ by first plotting some example points using $t = -2, -1, 0, 1, 2$. Write the curve as an equation of the form $y = f(x)$.

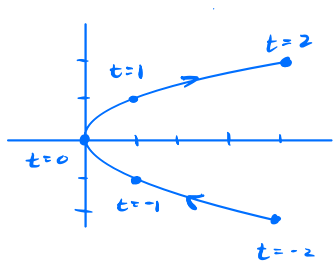
t	$\vec{r}(t)$
-2	$\langle -2, -8 \rangle$
-1	$\langle -1, -1 \rangle$
0	$\langle 0, 0 \rangle$
1	$\langle 1, 1 \rangle$
2	$\langle 2, 8 \rangle$



$x = t$, $y = t^3 \Rightarrow y = x^3$

Problem 4. Plot the curve given by $\mathbf{r}(t) = \langle t^2, t\rangle$ for $-2 \le t \le 2$ by first plotting some example points using $t = -2, -1, 0, 1, 2$. Write the curve as an equation involving the variables x and y.

t	$\vec{r}(t)$
-2	$\langle 4, -2\rangle$
-1	$\langle 1, -1\rangle$
0	$\langle 0, 0\rangle$
1	$\langle 1, 1\rangle$
2	$\langle 4, 2\rangle$



$$x = t^2, \quad y = t \implies x = y^2$$

Problem 5. Make a vector equation $\mathbf{r}(t)$ for a line that passes through the points $P = (1, 2, 6)$ and $Q = (-4, 3, 1)$ using the point-direction-vector form $\mathbf{p} + t\mathbf{v}$ we learned last week with $\mathbf{p} = \langle 1, 2, 6\rangle$. At what time t are you at the point P? Where are you at time $t = 1$? Explain how to make an equation so that you are at Q at time $t = 0$ and at P at time $t = 1$.

$$\vec{v} = \overrightarrow{PQ} = \langle -5, 1, -5\rangle$$

$$\vec{r}(t) = \langle 1, 2, 6\rangle + t\langle -5, 1, -5\rangle$$

$$\vec{r}(0) = P, \quad \vec{r}(1) = Q$$

if we use $\vec{s}(t) = \underbrace{\langle -4, 3, 1\rangle}_{Q} + t\,\underbrace{\langle 5, -1, 5\rangle}_{-\vec{v}}$

then $\vec{s}(0) = Q$ and $\vec{s}(1) = P$.

Problem 6. Use CalcPlot3D to sketch the following curves in $\mathbb{R}^3$.

a. $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{t}{6\pi} \right\rangle, 0 \le t \le 6\pi$

b. $\mathbf{r}(t) = \left\langle \cos t, \frac{t}{4\pi}, \sin t \right\rangle, 0 \le t \le 4\pi$