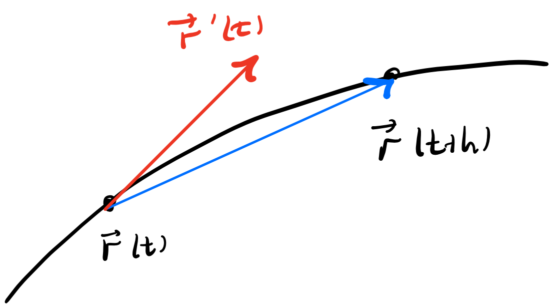
11.2 Calculus and Vector-Valued Functions

Given a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, it's derivative is also a vector-valued function:

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

(as long as f, g, h are differentiable).

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$



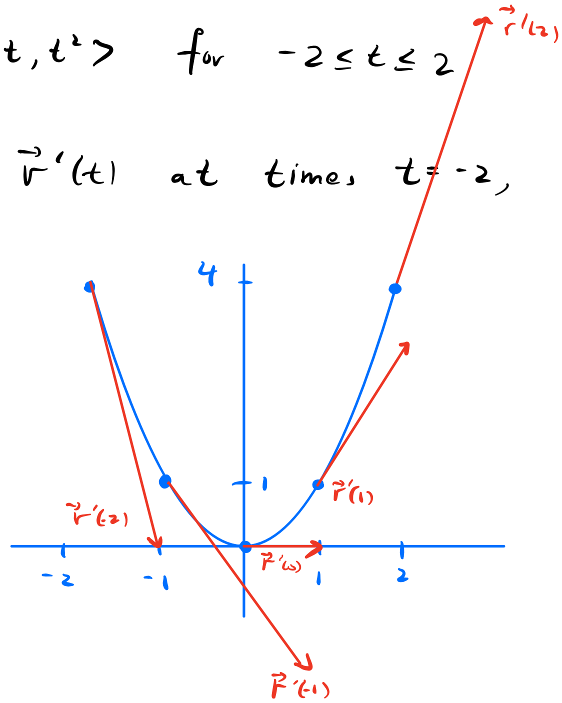
as $h \to 0$, the secant vector $\dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$ becomes tangent vector $\vec{r}\,'(t)$

Example Let $\vec{r}(t) = \langle t, t^2 \rangle$ for $-2 \le t \le 2$

Plot $\vec{r}(t)$ and plot $\vec{r}\,'(t)$ at times $t = -2$, -1, 0, 1, and 2.

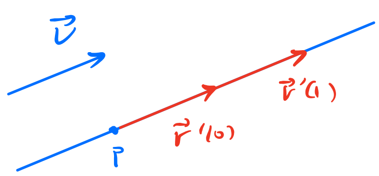
$\vec{r}\,'(t) = \langle 1, 2t \rangle$

t	$\vec{r}\,'(t)$
-2	$\langle 1, -4 \rangle$
-1	$\langle 1, -2 \rangle$
0	$\langle 1, 0 \rangle$
1	$\langle 1, 2 \rangle$
2	$\langle 1, 4 \rangle$



Example Let $\vec{r}(t) = \langle 1+2t, 4-3t, 5+7t \rangle$ be the line passing through $P = (1, 4, 5)$ in the direction of $\vec{v} = \langle 2, -3, 7 \rangle$. Compute $\vec{r}\,'(t)$.

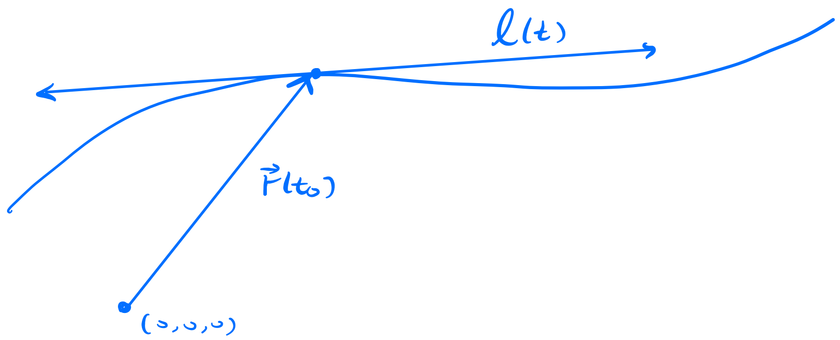
$\vec{r}\,'(t) = \langle 2, -3, 7 \rangle = \vec{v}$



$\vec{r}\,'(t)$ is constant

same tangent vector at every point along line

Tangent lines



Given curve $\vec{r}(t)$, we want to find a vector equation $\ell(t)$ for the tangent line to the curve at time $t=c$

point on line : $\vec{p} = \vec{r}(c)$

direction of line : $\vec{v} = \vec{r}\,'(c)$

$$\ell(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Example Find tangent line to $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at time $t = -1$.

$\vec{r}(-1) = \langle -1, 1, -1 \rangle$,

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$,

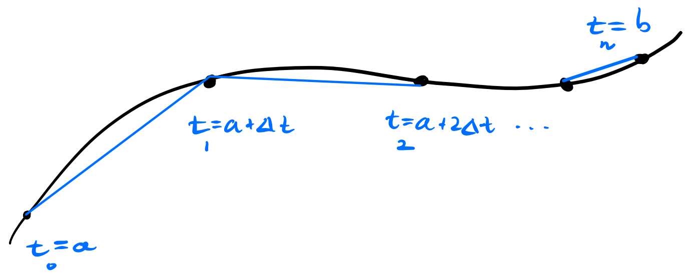
$\vec{r}\,'(-1) = \langle 1, -2, 3 \rangle$

$\ell(t) = \langle -1, 1, -1 \rangle + t\langle 1, -2, 3 \rangle$

Remark $\vec{r}\,'(t)$ represents velocity of particle whose position is given by $\vec{r}(t)$ at time t

$\|\vec{r}\,'(t)\|$ represents its speed at time t.

Arc length



The length of $\vec{r}(t)$ from $t=a$ to $t=b$

$$= \lim_{\Delta t \to 0} \|\vec{r}(t_1) - \vec{r}(t_0)\| + \|\vec{r}(t_2) - \vec{r}(t_1)\| + \cdots + \|\vec{r}(t_n) - \vec{r}(t_{n-1})\|$$

$$= \lim_{\Delta t \to 0} \frac{\|\vec{r}(t_1) - \vec{r}(t_0)\| + \|\vec{r}(t_2) - \vec{r}(t_1)\| + \cdots + \|\vec{r}(t_n) - \vec{r}(t_{n-1})\|}{\Delta t} \Delta t$$

$$= \lim_{\Delta t \to 0} \left(\frac{\|\vec{r}(t_1) - \vec{r}(t_0)\|}{\Delta t} + \cdots + \frac{\|\vec{r}(t_n) - \vec{r}(t_{n-1})\|}{\Delta t} \right) \Delta t$$

$$= \int_a^b \|\vec{r}\,'(t)\| \, dt$$

Formula Length of $\vec{r}(t)$ from $t=a$ to $t=b$ is given by $\int_a^b \|\vec{r}\,'(t)\| \, dt$.

Example Find the length of the helix

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad 0 \le t \le 4\pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{2}$$

So the length is

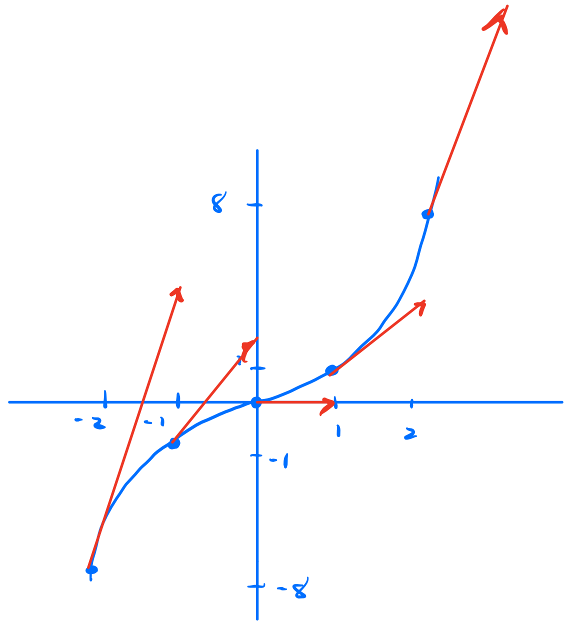
$$\int_0^{4\pi} \|\vec{r}\,'(t)\|\,dt = \int_0^{4\pi} \sqrt{2}\,dt$$

$$= 4\pi\sqrt{2}$$

Problem 1. Plot the curve given by $\mathbf{r}(t) = \langle t, t^3 \rangle$ for $-2 \le t \le 2$ and plot the tangent vectors $\mathbf{r}'(t)$ at times $t = -2, -1, 0, 1, 2$.

$\vec{r}\,'(t) = \langle 1, 3t^2 \rangle$

t	$\vec{r}(t)$	$\vec{r}\,'(t)$
-2	$\langle -2, -8 \rangle$	$\langle 1, 12 \rangle$
-1	$\langle -1, -1 \rangle$	$\langle 1, 3 \rangle$
0	$\langle 0, 0 \rangle$	$\langle 1, 0 \rangle$
1	$\langle 1, 1 \rangle$	$\langle 1, 3 \rangle$
2	$\langle 2, 8 \rangle$	$\langle 1, 12 \rangle$

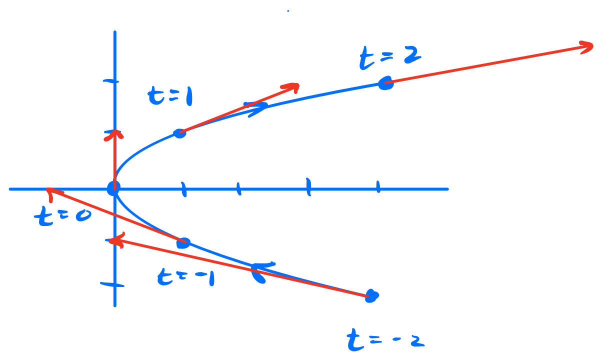


$x = t,\ y = t^3 \implies y = x^3$

Problem 2. Plot the curve given by $\mathbf{r}(t) = \langle t^2, t \rangle$ for $-2 \le t \le 2$ and plot the tangent vectors $\mathbf{r}'(t)$ at times $t = -2, -1, 0, 1, 2$.

$\vec{r}\,'(t) = \langle 2t, 1 \rangle$

t	$\vec{r}(t)$	$\vec{r}\,'(t)$
-2	$\langle 4, -2 \rangle$	$\langle -4, 1 \rangle$
-1	$\langle 1, -1 \rangle$	$\langle -2, 1 \rangle$
0	$\langle 0, 0 \rangle$	$\langle 0, 1 \rangle$
1	$\langle 1, 1 \rangle$	$\langle 2, 1 \rangle$
2	$\langle 4, 2 \rangle$	$\langle 4, 1 \rangle$



$x = t^2,\quad y = t \implies x = y^2$

Problem 3. Find a vector equation for the tangent line to curve $\mathbf{r}(t)$ at the given t value.

a. $\mathbf{r}(t) = \langle 3t^3 - 2t^2 + t + 1, t^4 + t^3 - 3t \rangle, t = 1$

b. $\mathbf{r}(t) = \left\langle 3\cos t, 3\sin t, \frac{t}{2\pi} \right\rangle, t = \pi$

(a) $\vec{r}\,'(t) = \langle 9t^2 - 4t + 1, 4t^3 + 3t^2 - 3 \rangle$

$\vec{r}(1) = \langle 3, -1 \rangle \qquad \vec{r}\,'(1) = \langle 6, 4 \rangle$

$\ell(t) = \langle 3, -1 \rangle + t \langle 6, 4 \rangle$

(b) $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, \frac{1}{2\pi} \rangle$

$\vec{r}(\pi) = \langle -3, 0, \frac{1}{2} \rangle, \quad \vec{r}\,'(\pi) = \langle 0, -3, \frac{1}{2\pi} \rangle$

$\ell(t) = \langle -3, 0, \frac{1}{2} \rangle + t \langle 0, -3, \frac{1}{2\pi} \rangle.$

Problem 4. Compute $\mathbf{r}'(t)$ for the following functions.

a. $\mathbf{r}(t) = \langle \cos t, e^t, \ln t \rangle$

b. $\mathbf{r}(t) = \left\langle \sin(2t), e^{t^2}, \cos(t^3) \right\rangle$ (remember the chain rule)

c. $\mathbf{r}(t) = \langle t\cos t, t^2 \sin t, t^3 \ln t \rangle$ (remember the product rule)

(a) $\vec{r}\,'(t) = \langle -\sin t, e^t, \frac{1}{t} \rangle$

(b) $\vec{r}\,'(t) = \langle 2\cos(2t), 2te^{t^2}, -3t^2 \sin(t^3) \rangle$

(c) $\vec{r}\,'(t) = \langle \cos t - t\sin t, 2t\sin t + t^2 \cos t, 3t^2 \ln t + t^2 \rangle$

Problem 5. Find the arc length of the following curves

a. $\mathbf{r}(t) = \langle t^3, 3t^2, 6t\rangle$, $0 \le t \le 4$

b. $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, 2t, \frac{4}{3}t^{3/2}\right\rangle$, $2 \le t \le 4$

c. $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 4t^{3/2}\rangle$, $1 \le t \le 2$

(a) $\vec{r}\,'(t) = \langle 3t^2, 6t, 6\rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{9t^4 + 36t^2 + 36} = 3\sqrt{t^4 + 4t^2 + 4} = 3\sqrt{(t^2+2)^2} = 3(t^2+2)$$

$$\int_0^4 \|\vec{r}\,'(t)\|\,dt = 3\int_0^4 (t^2+2)\,dt = 3\left(\frac{1}{3}t^3 + 2t \Big|_0^4\right) = 3\left(\frac{64}{3} + 8\right) = 64 + 24 = 88$$

(b) $\vec{r}\,'(t) = \langle t, 2, 2t^{1/2}\rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2 + 4 + 4t} = t + 2$$

$$\int_2^4 \|\vec{r}\,'(t)\|\,dt = \int_2^4 (t+2)\,dt = \frac{1}{2}t^2 + 2t \Big|_2^4 = (8+8) - (2+4) = 10$$

(c) $\vec{r}\,'(t) = \langle -2\sin(2t), 2\cos(2t), 6t^{1/2}\rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{4 + 36t}$$

$$\int_1^2 \|\vec{r}\,'(t)\|\,dt = \int_1^2 \sqrt{4+36t}\,dt$$

$u = 4 + 36t$

$du = 36\,dt$

$$= \frac{1}{36}\int_{40}^{76} u^{1/2}\,du$$

$$= \frac{1}{36}\left(\frac{2}{3}u^{3/2}\Big|_{40}^{76}\right) = \frac{1}{54}\left(76^{3/2} - 40^{3/2}\right)$$