Open sets

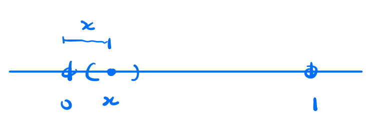
Def Let $A \subseteq \mathbb{R}$. Then A is called an open set if for every $x \in A$, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq A$.

Example The interval

$$(0,1) = \{ x \in \mathbb{R} : 0 < x < 1 \} \text{ is}$$

an open set. (We can generalize to any open interval)

Proof Let $x \in (0,1)$. We must show there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq (0,1)$. Consider the case when $x < \frac{1}{2}$. Define $\delta = \frac{x}{2}$.



We claim that $(x-\delta, x+\delta) \subseteq (0,1)$. Indeed, let $y \in (x-\delta, x+\delta)$. Then $y > x - \delta = x - \frac{x}{2} = \frac{1}{2}x > 0$.

Moreover, $y < x + \delta = x + \frac{x}{2} = \frac{3}{2}x < \frac{3}{4} < 1$.

Therefore $y \in (0,1)$ and so the desired inclusion holds.

Consider next the case when $x \geq \frac{1}{2}$. Define $\delta = \frac{1-x}{2}$.

Again we claim $(x-\delta, x+\delta) \subseteq (0,1)$. Let $y \in (x-\delta, x+\delta)$.

Then, $y > x - \delta = x - \frac{1-x}{2} = \frac{3}{2}x - \frac{1}{2} \geq \frac{3}{2}\left(\frac{1}{2}\right) - \frac{1}{2} = \frac{1}{4} > 0$.

Moreover, $y < x + \delta = x + \frac{1-x}{2} = \frac{1}{2}x + \frac{1}{2} < 1$ since $x < 1$.

Thus we again have the desired inclusion.

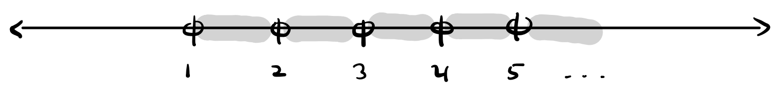
<u>Theorem</u> Let $\{A_\alpha : \alpha \in I\}$ be an arbitrary collection of open sets where I is a given index set. Then $\bigcup_{\alpha \in I} A_\alpha$ is open. In other words, any union of open sets is an open set.

<u>Proof</u> We must show that for any $x \in \bigcup_{\alpha \in I} A_\alpha$, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq \bigcup_{\alpha \in I} A_\alpha$.

Let $x \in \bigcup_{\alpha \in I} A_\alpha$. Then $x \in A_{\alpha_0}$ for some $\alpha_0 \in I$.

Since A_{α_0} is open, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq A_{\alpha_0}$. Since $A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$, we have $(x-\delta, x+\delta) \subseteq \bigcup_{\alpha \in I} A_\alpha$.

Example $\bigcup_{n=1}^{\infty} (n, n+1)$ is open



Since it is the union of open intervals which are open sets.

Problem 1. Make a conjecture about which of the following are open sets. No proof is needed for now, but I'll ask you to prove your conjecture for some of these on homework.

a. $\emptyset$

b. $\mathbb{R}$

c. $\{1\}$

d. $[0, 1)$

e. $\bigcap_{n=1}^{10}(1 - 1/n, 1 + 1/n)$

f. $\bigcap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$

(a) open

(b) open

(c) not open

(d) not open

(e) $(0,2) \cap (\frac{1}{2}, \frac{3}{2}) \cap (\frac{2}{3}, \frac{4}{3}) \cap (\frac{3}{4}, \frac{5}{4}) \cdots \cap (\frac{9}{10}, \frac{11}{10})$

$= (\frac{9}{10}, \frac{11}{10})$ open

(f) $\bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n}) = \{1\}$ not open

Problem 2. Prove or disprove: the intersection of a finite collection of open sets is open.

Proof Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a finite collection open sets. We claim $B = \bigcap_{i=1}^{n} A_i$ is open. Let $x \in B$. We must show there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq B$. Observe that since $x \in B$, $x \in A_i$ for all $i = 1, \dots, n$. Therefore for each $i = 1, \dots, n$ there exists δ_i such that $(x - \delta_i, x + \delta_i) \subseteq A_i$. Let $\delta = \min\{\delta_1, \dots, \delta_n\}$. Then for all $i = 1, \dots, n$

$$(x-\delta, x+\delta) \subseteq (x - \delta_i, x + \delta_i) \subseteq A_i,$$

which implies $(x-\delta, x+\delta) \subseteq \bigcap_{i=1}^{n} A_i = B$.

Problem 3. Prove or disprove: the intersection of an infinite collection of open sets is open.

Counterexample part f of Problem 1.