Open and Closed Sets Wrap up

Theorem Let I be a given index set and $\{A_\alpha : \alpha \in I\}$ a collection of open sets and $\{B_\alpha : \alpha \in I\}$ a collection of closed sets. Then

$$\bigcup_{\alpha \in I} A_\alpha \text{ is open and } \bigcap_{\alpha \in I} B_\alpha \text{ is closed.}$$

Proof We've previously proved $\bigcup_{\alpha \in I} A_\alpha$ is open.

To show $\bigcap_{\alpha \in I} B_\alpha$ is closed we must show $\left(\bigcap_{\alpha \in I} B_\alpha\right)^c$ is open. Observe that by De Morgan's Law, $\left(\bigcap_{\alpha \in I} B_\alpha\right)^c = \bigcup_{\alpha \in I} B_\alpha^c$. Notice that since B_α is closed for all $\alpha \in I$, B_α^c is open for all $\alpha \in I$.

Therefore $\bigcup_{\alpha \in I} B_\alpha^c$ is open.

Theorem Let $m \in \mathbb{Z}^+$ and consider a finite collection $\{A_1, \ldots, A_m\}$ of open sets. Then $\bigcap_{n=1}^{m} A_n$ is open.

Proof Let $x \in \bigcap_{n=1}^{m} A_n$. We must show there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq \bigcap_{n=1}^{m} A_n$. Notice that since $x \in A_n$ and A_n is open for each $n = 1, \ldots, m$, there exist $\delta_1, \ldots, \delta_m > 0$ such that $(x - \delta_n, x + \delta_n) \subseteq A_n$ for each $n = 1, \ldots, m$.

Let $\delta = \min\{\delta_1, \ldots, \delta_m\}$. We claim $(x-\delta, x+\delta) \subseteq \bigcap_{n=1}^{m} A_n$. Let $y \in (x-\delta, x+\delta)$. Notice since $\delta < \delta_n$ we have

$$(x-\delta, x+\delta) \subseteq (x-\delta_n, x+\delta_n) \subseteq A_n$$

for all $n = 1, \ldots, m$. Therefore $y \in A_n$ for all $n = 1, \ldots, m$ and so $y \in \bigcap_{n=1}^{m} A_n$ and our claim is proved.

Fact If $\{A_\alpha : \alpha \in I\}$ is an infinite collection of open sets, $\bigcap_{\alpha \in I} A_\alpha$ might be open, closed, or neither.

Examples

$A_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$, $\bigcap_{n=1}^{\infty} A_n = \{1\}$ is closed.

$B_n = \left(0, 1 + \frac{1}{n}\right)$, $\bigcap_{n=1}^{\infty} B_n = (0, 1]$ is neither open nor closed

$C_n = (n, n+1)$, $\bigcap_{n=1}^{\infty} C_n = \emptyset$ is open (and closed)

Fact If $\{A_\alpha : \alpha \in I\}$ is an infinite collection closed sets, $\bigcup_{\alpha \in I} A_\alpha$ might be open, closed, or neither.

Let $A_n = (-\infty, 1-\frac{1}{n}] \cup [1+\frac{1}{n}, \infty)$

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c = \bigcap_{n=1}^{\infty} (1-\tfrac{1}{n}, 1+\tfrac{1}{n}) = \{1\}$$

and so $\bigcup_{n=1}^{\infty} A_n = (-\infty, 1) \cup (1, \infty)$ is open.

$B_n = [0, n]$, $\bigcup_{n=1}^{\infty} B_n = [0, \infty)$ is closed

$$C_n = (-\infty, 0] \cup [1+\tfrac{1}{n}, \infty), \left(\bigcup_{n=1}^{\infty} C_n\right)^c = \bigcap_{n=1}^{\infty} C_n^c$$

$$= \bigcap_{n=1}^{\infty} (0, 1+\tfrac{1}{n})$$

$$= (0, 1]$$

So $\bigcup_{n=1}^{\infty} C_n = (-\infty, 0] \cup (1, \infty)$ is neither open nor closed.