Chapter 10 Relations

Def Let X be a given set. A relation on X is a subset $S \subseteq X \times X$. In the context of relations, if $(x,y) \in S$, we denote this as $x \sim y$ and say "x is related to y."

we will denote this as $\mathbb{R}^2$ from now on

Example 1 Let $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x > y\}$

Then $2 \sim 1$ but $1 \nsim 2$ since $(2,1) \in S$ but $(1,2) \notin S$.

Example 2 Let X be a given set and $\mathcal{P}(X)$ its power set. We can define a relation on $\mathcal{P}(X)$ by saying, given $A, B \in \mathcal{P}(X)$, $A \sim B$ if $A \subseteq B$. In the set notation, this relation is the set $S = \{(A,B) \in \mathcal{P}(A) \times \mathcal{P}(A) : A \subseteq B\}$.

Example 3 Consider the relation on $\mathbb{Z}$ given by $S = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : 3 \mid (m-n)\}$.

Is $1 \sim 2$? (no) $1 \sim 3$? (no) $1 \sim 4$? (yes) List all $n \in \mathbb{Z}$ such that $1 \sim n$. $\{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$

Example 4 Consider the relation on $\mathbb{R}$ given by

$x \sim y$ if and only if $x - y \in \mathbb{Z}$.

① List all $x \in \mathbb{R}$ such that $3 \sim x$. $\mathbb{Z}$

② Is $-3.7 \sim 3.7$? No, since $-3.7 - 3.7 = -7.4 \notin \mathbb{Z}$.

③ List all $x \in \mathbb{R}$ such that $1.4 \sim x$

$$\{ \ldots, -2.6, -1.6, -0.6, 0.4, 1.4, 2.4, 3.4, \ldots \}$$

Def An equivalence relation $\sim$ on X is a relation on X that is

① reflexive: $\forall\, x \in X$, $x \sim x$

② symmetric $\forall\, x, y \in X$, if $x \sim y$, then $y \sim x$.

③ transitive $\forall\, x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Exercise Which examples above are equivalence relations? For those that are not, what fails?

① Example 1: no. reflexivity and symmetry fail

② Example 2: no, symmetry fails

(3) Example 3 : yes

Proof (1) Let $n \in \mathbb{Z}$. Then since $3 \cdot 0 = 0$ and $0 \in \mathbb{Z}$, $3 \mid 0$, and so $3 \mid (n-n)$. Thus $n \sim n$.

(2) Suppose $n \sim m$. Then $3k = n - m$ for some $k \in \mathbb{Z}$. This implies $3(-k) = m - n$.

Since $-k \in \mathbb{Z}$, $m \sim n$.

(3) Suppose $n \sim m$ and $m \sim p$. Then

$3j = (n-m)$ and $3k = (m-p)$ for some $j, k \in \mathbb{Z}$.

Therefore
$$n - p = n - m + m - p$$
$$= 3j + 3k = 3(j+k).$$

Since $j+k \in \mathbb{Z}$, $3 \mid (n-p)$ and so $n \sim p$.

(4) Example 4 : yes

Proof (1) Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$. Thus $x \sim x$.

(2) Suppose $x \sim y$. Then $x - y = k$ for some $k \in \mathbb{Z}$. Therefore $y - x = -k$.

Since $-k \in \mathbb{Z}$, $y \sim x$.

(3) Suppose $x \sim y$ and $y \sim z$. Then

$x - y = k$ and $y - z = j$ for some $j, k \in \mathbb{Z}$. Therefore $x - z = x - y + y - z = k + j$.

Since $k + j \in \mathbb{Z}$, $x \sim z$.

Def Given an equivalence relation $\sim$ on X, for each $x \in X$, the equivalence class of x is the set $E_x \subseteq X$ given by $E_x = \{ y \in X : y \sim x \}$.

Examples In Example 3 above

$E_0 = \{ 3n : n \in \mathbb{Z} \}$, $E_1 = \{ 3n+1 : n \in \mathbb{Z} \}$.

In Example 4 above,

$E_0 = \mathbb{Z}$, $E_{0.2} = \{ 0.2 + n : n \in \mathbb{Z} \}$.

Problem 1. Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x^2 + y^2$ is even. Show that this is an equivalence relation. Give 5 examples of integers in the equivalence class of 1.

Let $x, y, z \in \mathbb{Z}$. Note that $x \sim x$ since $x^2 + x^2 = 2x^2$ is even. Suppose $x \sim y$. Then $x^2 + y^2$ is even. This also implies $y^2 + x^2$ is even and so $y \sim x$. Suppose $x \sim y$ and $y \sim z$. Then $x^2 + y^2$ and $y^2 + z^2$ are even. That $x^2 + y^2 = 2n$ and $y^2 + z^2 = 2m$ for some $n, m \in \mathbb{Z}$. Therefore $x^2 + z^2 = 2n - y^2 + 2m - y^2 = 2(m - y^2)$, which implies $x^2 + z^2$ is even since $m - y^2 \in \mathbb{Z}$. Thus $x \sim z$.

$E_1 = \{ 2n+1 : n \in \mathbb{Z} \}$.

Problem 2. Consider the relation on $\mathbb{Z}$ given by $x \sim y$ if and only if $y - x$ is divisible by 7. It is possible to show that this is an equivalence relation. For each of $x = 0, 1, 2$ find give elements of E_x.

$$E_0 = \{7n : n \in \mathbb{Z}\} = \{\ldots, -21, -14, -7, 0, 7, 14, 21, \ldots\}$$

$$E_1 = \{7n+1 : n \in \mathbb{Z}\} = \{-20, -13, -6, 1, 8, 15, 22, \ldots\}$$

$$E_2 = \{7n+2 : n \in \mathbb{Z}\} = \{-19, -12, -5, 2, 9, 16, 23, \ldots\}$$

Problem 3. Define a relation on $\mathbb{R}$ by $x \sim y$ if and only if there exists $n \in \mathbb{Z}$ such that $x, y \in [n, n+1]$. Is this an equivalence relation? Why or why not?

No. Notice that if $x = 0$, $y = 1$, $z = 2$ then

$x \sim y$ and $y \sim z$ but $x \not\sim z$.

Problem 4. Consider the following modification of the relation in the previous example. Define a relation on $\mathbb{R}$ by $x \sim y$ if and only if there exists $n \in \mathbb{Z}$ such that $x, y \in [n, n+1)$. This is in fact an equivalence relation. (Try proving it if you finish early.) What elements are in the equivalence class E_π? What elements are in the equivalence class of -1?

$$E_\pi = [3, 4)$$

$$E_{-1} = [-1, 0)$$