Chapter 12 Order in the reals, part II

We saw that $\{x \in \mathbb{Q} : x^2 \leq 2\} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$

is an example of a set that had no min/max.

Moreover, the supremum and infimum, while they exist, are not in the set because they're not rational.

This is the key property that distinguishes the real numbers from the rationals.

__Completeness Axiom of the Reals__ Every nonempty subset of real numbers that is bounded above has a supremum.

In other words, the reals are a number system that is closed under taking supremum of sets that are bounded above.

__Theorem__ (Archimedean property of $\mathbb{R}$) Let $a, b \in (0, \infty)$. Then there exists $n \in \mathbb{Z}^+$ such that $nb > a$.

__Proof__ Suppose by way of contradiction this is not true.

Then there exist $a, b \in (0, \infty)$ such that for all $n \in \mathbb{Z}^+$ $nb \leq a$. Then a is an upper bound for the set $S = \{nb : n \in \mathbb{Z}^+\}$. Since S is bounded above, it has a supremum $u = \sup S$ by the Completeness Axiom. Consider $u-b$. Since $u - b < u$ and u is the least upper bound, $u - b$ cannot be an upper bound of S. So there exists some $n_0 \in \mathbb{Z}^+$ such that $n_0 b > u - b$. This implies $(n_0+1)b > u$. But $(n_0+1)b \in S$, so this contradicts that u is an upper bound of S.

__Corollary__ For every $x \in (0, \infty)$, there exists $n \in \mathbb{Z}^+$ such that $x < n$. Moreover, there exists $m \in \mathbb{Z}^+$ such that $\frac{1}{m} < x$.

__Proof__ For the first part, use the Archimedean property with $a = x$, $b = 1$. For the second part, use $a = 1$, $b = x$.

__Well-ordering principle of $\mathbb{N}$__ Every nonempty subset of $\mathbb{N}$ has a minimum. (An axiom we __assume__ about $\mathbb{N}$)

Theorem Let $a, b \in \mathbb{R}$ such that $a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof We'll do the case when $a, b \in (0, \infty)$ and leave the other cases as an exercise.

By the corollary of the Archimedean property there exists $m \in \mathbb{Z}^+$ such that $\frac{1}{m} < b - a$.

Consider the set $S = \{ n \in \mathbb{N} : \frac{n}{m} > a \}$. By the corollary of the Archimedean property S is nonempty. By the well-ordering principle of $\mathbb{N}$, S has a minimum, $n_0 = \min S$. We claim $\frac{n_0}{m} < b$. Notice since $n_0 - 1 < n_0$ and $n_0 = \min S$, $\frac{n_0 - 1}{m} \leq a$ which implies $\frac{n_0}{m} \leq a + \frac{1}{m} < b$.

Problem 1. Let $S = \mathbb{Q} \cap (0, 4)$. Consider the following incorrect proof that $\sup S = 4$. What is the logical flaw with this proof?

Proof. First we show 4 is an upper bound. Let $x \in S$. Then $0 < x < 4$, so 4 is clearly an upper bound. Suppose to the contrary that 4 is not the supremum. Then there exists an upper bound u with $u < 4$. But $u < (u + 4)/2$ and we have shown that u is not an upper bound of S. This shows that 4 must be the supremum. □

The flaw is that it's not necessarily the case that $\frac{u+4}{2} \in S$ since we don't know $u \in S$. That means $u < \frac{u+4}{2}$ doesn't necessarily contradict u is an upper bound of S.

Problem 2. Fix the proof above, taking advantage of the fact that between any two real numbers there is a rational number.

Let $x \in S$. Then $0 < x < 4$, so 4 is an upper bound of S. Suppose u is an upper bound of S. We claim $u \geq 4$. Suppose not. Then $u < 4$. Since $1 \in S$ and u is an upper bound of S, $1 \leq u$. Thus $u \in (0, 4)$. Moreover, there exists $q \in \mathbb{Q}$ such that $0 < u < q < 4$. Thus $q \in S$ and $u < q$ which contradicts that u is an upper bound of S. Thus $u \geq 4$ and so 4 is the least upper bound of S.

Problem 3. Let $S = \{1 - 1/n : n \in \mathbb{Z}^+\}$. Show that $\sup S = 1$.

Observe that $1 - \frac{1}{n} \leq 1$ for all $n \in \mathbb{Z}^+$. Thus 1 is an upper bound of S. Let u be an upper bound of S. We claim $u \geq 1$. Suppose not. Then $u < 1$. By the Archimedean property, there exists $m \in \mathbb{Z}^+$ such that $\frac{1}{m} < 1 - u$. But this implies $u < 1 - \frac{1}{m}$ which contradicts that u is an upper bound of S.