Chapter 13 More on completeness.

Def A set $A \subseteq \mathbb{R}$ is said to be complete if, for any subset $B \subseteq A$ that is nonempty and bounded above, $\sup B \in A$.

Theorem The set $\mathbb{Q}$ is not complete. That is there exists a nonempty set $B \subseteq \mathbb{Q}$ that is bounded above but there does not exist $U \in \mathbb{Q}$ such that

① $x \le U \quad \forall x \in B$

② if $M \in \mathbb{Q}$ is an upper bound of B, then $M \ge U$.

Proof Consider $B = \{x \in \mathbb{Q}^+ : x^2 < 2\}$. Since $1 \in B$ and 2 is an upper bound of B, B is nonempty and bounded above. Suppose by of contradiction that there exist $U \in \mathbb{Q}$ satisfying the conditions above. Then $U = \sqrt{2}$ or $U > \sqrt{2}$ or $U < \sqrt{2}$. Since $\sqrt{2} \notin \mathbb{Q}$ the first case is impossible. Suppose $U > \sqrt{2}$. Then there exists a rational number q such that $\sqrt{2} < q < U$. Since $\sqrt{2} < q$, for every $x \in B$, $x < \sqrt{2} < q$. So q is an upper bound of B, but $q < U$ contradicts ②.

Suppose $U < \sqrt{2}$. Then there exists $q \in \mathbb{Q}$ such that $U < q < \sqrt{2}$. Then $q < \sqrt{2}$ implies $q^2 < 2$ which implies $q \in B$. But then $U < q$ contradicts ①.

Problem 1. Consider the set $S = \mathbb{R} \setminus \mathbb{Z}$. Give an example of a subset $T_1 \subseteq S$ that has an infimum and supremum in S and a bounded subset $T_2 \subseteq S$ that has neither an infimum nor supremum in S. *Note that the existence of such a set T_2 shows that S is not complete.*

$T_1 = [0.1, 0.9]$

$T_2 = (0, 1)$

Problem 2. Determine whether each of the following is true or false.

a. Let S be a subset of $\mathbb{R}$ consisting of 20 positive integers. Then S has a supremum and an infimum, both of which belong to S.

b. Suppose that S is a nonempty subset of $\mathbb{R}$ and S has a supremum U. Let $T = \{x \in S : x \leq U\}$. Then $T = S$.

c. Suppose that S is a nonempty bounded subset of $\mathbb{R}$ and $U = \sup S$. Suppose further that there exists $x \in S$ such that $x < U$. Let $v = \sup\{x \in S : x < U\}$. Then $v < U$.

(a) true

(b) true:

Proof It is clear that $T \subseteq S$ since T consists o elements of S under some condition. We must show $S \subseteq T$. Let $x \in S$. Then $x \leq U$ since U is an upper bound of S. Therefore $x \in T$. Thus $S \subseteq T$ and so $S = T$.

(c) false